Ann Glob Anal Geom (2007) 31:287–328
DOI 10.1007/s10455-007-9061-0

ORIGINAL PAPER

Group actions on chains of Banach manifolds and applications to fluid dynamics

François Gay-Balmaz · Tudor S. Ratiu

Received: 28 March 2006 / Accepted: 9 June 2006 / Published online: 24 February 2007

Abstract This paper presents the theory of non-smooth Lie group actions on chains of Banach manifolds. The rigorous functional analytic spaces are given to deal with quotients of such actions. A hydrodynamical example is studied in detail.

Keywords Banach manifold · Reduction · Lie group action · Slices · Euler equation

AMS Classification 58B20 · 58B25 · 58D05 · 58D19 · 35Q35 · 53D1 · 53D25

1 Introduction

The goal of this paper is to show the existence of *slices*, to study the geometric properties of the *orbit type sets*, the *fixed point sets*, the *isotropy type sets*, and the *orbit spaces* associated to a certain important class of non-smooth actions of a Lie group G on a chain of Banach manifolds. These considerations will be applied to the motion of an ideal homogeneous incompressible fluid in a compact domain.

This study is motivated by many examples that appear as results of various reduction procedures of infinite dimensional Hamiltonian systems with symmetry. All these applications have two technical difficulties in common. First, the evolutionary equations do not have smooth flows in the function spaces that are natural to the problem. If the system is linear, this corresponds to the fact that the equation is defined by an unbounded operator. This leads to the study of vector fields on manifolds of maps that

F. Gay-Balmaz (✉) · T. S. Ratiu
Section de Mathématiques,
École Polytechnique Fédérale de Lausanne,
1015 Lausanne, Switzerland
e-mail: Francois.Gay-Balmaz@epfl.ch

T. S. Ratiu
e-mail: Tudor.Ratiu@epfl.ch

are defined only on open dense subsets (see Ref. [3] for a presentation of this theory). Second, the actions, even those of finite dimensional Lie groups, are not smooth in the usual sense. The dependence on the group variable is only smooth on an open dense set. It turns out that in spite of these problems one can endow various objects related to the symmetry of the manifold and the flows of various interesting vector fields with certain weak smooth structures that are compatible between themselves. It is the goal of this paper to present the beginnings of such a theory and work out a concrete example coming from fluid dynamics.

Before presenting the outline of this theory and discussing some applications let us recall briefly some standard results about the structure of orbit spaces in the finite dimensional case. Consider a smooth and proper action

$$\Phi\colon G \times M \longrightarrow M$$

of a finite dimensional Lie group G on a smooth finite dimensional manifold M. Then we have the following results.

(i) If the action is free, M/G is a smooth manifold and the canonical projection $\pi : M \longrightarrow M/G$ defines on M the structure of a smooth principal G-bundle.

(ii) If all the isotropy subgroups are conjugate to a given one, say $H \subset G$, then M/G is a smooth manifold and the canonical projection $\pi : M \longrightarrow M/G$ defines on M the structure of a smooth locally trivial fiber bundle with structure group $N(H)/H$ and fiber G/H, where

$$N(H) := \{g \in G \mid gHg^{-1} = H\}$$

is the normalizer of H in G.

(iii) For an arbitrary smooth proper action, if $H \subset G$ is a closed subgroup of G, define the following subsets of M:

$$\begin{aligned} M_{(H)} &:= \left\{m \in M \mid G_m \text{ is conjugate to } H\right\}, \\ M^H &:= \{m \in M \mid H \subset G_m\}, \\ M_H &:= \{m \in M \mid G_m = H\}. \end{aligned}$$

$M_{(H)}$ is called the (H)-*orbit type set*, M^H is the H-*fixed point set*, and M_H is the H-*isotropy type set*. $M_{(H)}$ is G-invariant whereas M_H and M^H are not G-invariant, in general. In addition, $M_H \subset M^H$.

(iv) $M_{(H)}, M^H$, and M_H are submanifolds of M. Moreover, M_H is open in M^H and for $m \in M^H$,

$$T_mM^H = \{v_m \in T_mM \mid T_m\Phi_h(v_m) = v_m, \forall\, h \in H\}.$$

We have the partitions

$$M = \bigsqcup_{(H)} M_{(H)} \quad \text{and} \quad M/G = \bigsqcup_{(H)} (M_{(H)}/G);$$

$M_{(H)}/G$ is called the *isotropy stratum of type* (H). By (ii), $\pi_{(H)} := \pi|_{(H)} : M_{(H)} \longrightarrow M_{(H)}/G$ is a smooth locally trivial fiber bundle with structure group $N(H)/H$ and fiber G/H.

Of course (i) is a particular case of (ii), and (ii) is a particular case of (iii)–(iv). With the additional hypothesis that the orbit map $\Phi^m\colon G \to M$ is an immersion for all $m \in M$, the result in (i) is still valid for Banach Lie groups acting smoothly and properly on Banach manifolds (see Ref. [2], Ch. III, Sect. 1, Proposition 10) . This hypothesis always holds in finite dimensions.

We now present some examples of G-actions on Banach manifolds that arise in the reduction by stages procedure of infinite dimensional dynamics. We shall see that the previous results are not applicable to these actions since they are not smooth.

1.1 First example: incompressible fluid dynamics

Let (M,g) be a smooth Riemannian manifold and denote by $Iso := Iso(M,g)$ the finite dimensional Lie group of isometries of (M,g) (the Meyers and Steenrod [13] Theorem). The Lie group topology of Iso coincides with the topology of uniform convergence on compact sets. If M is compact, the Lie group Iso, and hence Iso^+, is a compact Lie group. For the proof of these statements see Kobayashi and Nomizu [10], Theorem 3.4 in Chapter VI.

We consider the motion of an ideal incompressible fluid in a compact oriented Riemannian manifold M with boundary. The appropriate configuration space is $\mathcal{D}^s_\mu(M), s > \frac{\dim M}{2}+1$, the Hilbert manifold of volume preserving H^s-diffeomorphisms of M. The Lagrangian is the quadratic form associated to the weak L^2 Riemannian metric on $\mathcal{D}^s_\mu(M)$ given by

$$\langle\langle u_\eta, v_\eta\rangle\rangle = \int_M g(\eta(x))(u_\eta(x), v_\eta(x))\mu(x), \quad u_\eta, v_\eta \in T_\eta\mathcal{D}^s_\mu(M),$$

where g is the Riemannian metric and μ is the associated Riemannian volume form.

Since this Lagrangian is invariant under the following two commuting actions

$$R : \mathcal{D}^s_\mu(M) \times T\mathcal{D}^s_\mu(M) \longrightarrow T\mathcal{D}^s_\mu(M), \;\; R_\eta(v_\xi) = v_\xi \circ \eta \;\; \text{and}$$

$$L : Iso^+ \times T\mathcal{D}^s_\mu(M) \longrightarrow T\mathcal{D}^s_\mu(M), \;\; L_i(v_\xi) = Ti \circ v_\xi,$$

it is formally true that the Poisson reduction by stages procedure can be applied, that is, the reduction by $\mathcal{D}^s_\mu(M)$ and then by Iso^+ coincides with the one step reduction by the product group $\mathcal{D}^s_\mu(M) \times Iso^+$.

The reduction by $\mathcal{D}^s_\mu(M)$ (first stage reduction) is well known and leads to the Euler equations for an ideal incompressible fluid on the first reduced space $\mathfrak{X}^s_{div}(M) = T\mathcal{D}^s_\mu(M)/\mathcal{D}^s_\mu(M)$.

Our goal is to carry out in a precise sense the reduction by Iso^+ (second stage reduction). In spite of the fact that Iso^+ is a compact finite dimensional Lie group, several problems occur. The action l of Iso^+ on $\mathfrak{X}^s_{div}(M)$ induced by L is given by

$$l : Iso^+ \times \mathfrak{X}^s_{div}(M) \longrightarrow \mathfrak{X}^s_{div}(M), \;\; l_i(u) = Ti \circ u \circ i^{-1} = i_*u$$

and we remark that l is neither free nor C^1 (l is C^1 as a map with values in $\mathfrak{X}^{s-1}_{div}(M)$). Thus the usual results about the geometric properties of the isotropy type submanifolds and the orbit space, valid for the case of a smooth proper action, cannot be used here. So it would be interesting to determine the exact differentiable structure of $\mathfrak{X}^s_{div}(M)_H$, $\mathfrak{X}^s_{div}(M)_H/N(H)$, $\mathfrak{X}^s_{div}(M)_{(H)}$, $\mathfrak{X}^s_{div}(M)_{(H)}/Iso^+$ in order to define tangent bundles, vector fields, evolution equations on them, and to carry out the second stage reduction procedure.

The following relevant facts are a guide in the search of a useful definition for the tangent bundle of the orbit spaces:

(1) the infinitesimal generators of the action l are not vector fields on $\mathfrak{X}^s_{div}(M)$ but they are sections of the vector bundle $\mathfrak{X}^s_{div}(M) \times \mathfrak{X}^{s-1}_{div}(M) \longrightarrow \mathfrak{X}^s_{div}(M)$,
(2) the Hamiltonian vector field is given by $X_h(u) = (u, -\mathrm{P}_e(\nabla_u u))$, so it is not a vector field on $\mathfrak{X}^s_{div}(M)$, since it takes values in $\mathfrak{X}^s_{div}(M) \times \mathfrak{X}^{s-1}_{div}(M)$,
(3) the integral curves of the Euler equation are not C^1 curves but are elements of $C^0(I, \mathfrak{X}^s_{div}(M)) \cap C^1(I, \mathfrak{X}^{s-1}_{div}(M))$.

We will treat this example in detail in Sect. 6. Note that the same situation arises in the motion of the averaged incompressible fluid.

1.2 Second example: a nonlinear wave equation

On the configuration space $H^s(S^1, \mathbb{R}^2) \times H^{s-1}(S^1, \mathbb{R}^2), s \geq 2$, we consider the Lagrangian

$$L(\varphi, \dot{\varphi}) = \frac{1}{2}\langle \dot{\varphi}, \dot{\varphi} \rangle - \frac{1}{2}\langle \varphi', \varphi' \rangle + \frac{1}{4}\langle \varphi, \varphi \rangle^2$$

where $\langle \, , \rangle$ is the L^2 inner product, the dot denotes time derivative, and the prime space derivative. Then the integral curves $\varphi(t)$ of the corresponding Euler–Lagrange equation are in fact periodic solutions of the nonlinear wave equation

$$\ddot{\varphi}(t) = \varphi(t)'' + \varphi(t)|\varphi(t)|^2.$$

The lagrangian L is invariant under the two following commuting actions:

$$L\colon SO(2) \times (H^s \times H^{s-1}) \longrightarrow H^s \times H^{s-1}, \ \ L_A(\varphi, \psi)(s) = (A \cdot \varphi(s), A \cdot \psi(s))$$

$$R\colon S^1 \times (H^s \times H^{s-1}) \longrightarrow H^s \times H^{s-1}, \ \ R_\alpha(\varphi, \psi)(s) = (\varphi(s + \alpha), \psi(s + \alpha)).$$

As before, it should be formally possible to apply the Poisson reduction by stages. Since the action of $SO(2)$ on $P^s := H^s \times H^{s-1} - \{(0,0)\}$ is smooth free and proper, the first reduced space $P^s/SO(2)$ is a Hilbert manifold (see [2], Chapter III, Proposition 10).

Some problems occur for the second reduced space. In fact, as was the case in the first example, the action of S^1 on $P^s/SO(2)$ induced by R is neither free nor C^1. The same problems arise in the symplectic reduction by stages where S^1 acts on $\mathbf{J}^{-1}_{SO(2)}(\mu)/SO(2)$ or on $\mathbf{J}^{-1}_{\mu,S^1}(\nu)$. Here $\mathbf{J}_{SO(2)}, \mathbf{J}_{S^1}$ denote the momentum mappings for the corresponding actions, and $\mathbf{J}_{\mu,S^1}$ denotes the map induced by $\mathbf{J}_{S^1}$ on $\mathbf{J}^{-1}_{SO(2)}(\mu)/SO(2)$.

In order to solve these two problems simultaneously, we will define, in Sect. 2, a precise notion of a non-smooth action of a finite dimensional Lie group on a collection of smooth Banach manifolds. We will see that the actions that appear in the previous examples are particular cases of these non-smooth actions. In Sect. 3 we shall show that the non-smooth actions we consider admit slices. This fact will be useful in the study of the geometric properties of the orbit type sets, the isotropy type sets, the fixed point sets, and of the orbit space. Due to the fact that the action is not smooth, the isotropy strata are not smooth manifolds. However, we will prove in Sect. 4 that, in the case all the isotropy groups are conjugated, the orbit space is a topological Banach manifold, by constructing explicit charts. Then we shall prove that the changes of charts are C^1 with respect to a weaker topology. This will allow us to define in Sect. 5 a weak tangent bundle for the orbit space, the notion of weak differentiable curves, as well as the notion of differentiable functions on the orbit space. The goal of Sect. 6 is to apply all the results of this paper to the case of the motion of the incompressible fluid in order to carry out in a precise sense the Poisson reduction by stages relative to the commuting actions R and L given in the first example.

2 Non-smooth actions and their orbits

Let $\{Q^s | s > s_0\}$ be a collection of smooth Banach manifolds such that for all $r > s > s_0$ there is a smooth inclusion $j_{(r,s)} : Q^r \hookrightarrow Q^s$ with dense range satisfying the following condition: for all $q \in Q^r$, the range of the tangent map $T_q j_{(r,s)} : T_q Q^r \longrightarrow T_q Q^s$ is dense. Density in these two conditions is always relative to the ambient spaces Q^s and $T_q Q^s$, respectively. In addition, we suppose that for each chart $\varphi^s : U^s \longrightarrow T_q Q^s$ of Q^s at $q \in Q^r, r > s$, the map $\varphi^r := \varphi^s | U^s \cap Q^r$ takes values in $T_q Q^r$ and is a chart for Q^r. If these hypotheses hold, $\{Q^s \mid s > s_0\}$ is called a *chain of Banach manifolds*.

Typical examples of chains of Banach manifolds are the collections of manifolds of maps $Q^s := H^s(M, N), s > \frac{\dim M}{2}$, or $Q^s := C^s(M, N), s \geq 1$, where M and N are compact and oriented finite dimensional manifolds (M possibly with boundary).

We suppose that each $Q^s, s > s_0$ carries a weak Riemannian metric γ_s which does not depend on s, that is $j^*_{(r,s)} \gamma_s = \gamma_r$. So we will suppress the index s and this metric will be denoted simply by γ.

For example, on $H^s(M, N)$ or $C^s(M, N)$ we can consider the L^2 metric

$$\gamma(f)(u_f, v_f) := \int_M g(f(x))(u_f(x), v_f(x))\mu(x)$$

where g is a Riemannian metric on N and μ is a volume form on M.

Let G be a finite dimensional Lie group. We suppose that for all $s > s_0$ we have a *proper* and *continuous* action

$$\Phi : G \times Q^s \longrightarrow Q^s, \quad \Phi(g, q) = \Phi_g(q) = \Phi^q(g)$$

of G on Q^s *compatible* with the restrictions $j^*_{(r,s)}$, that is,

- Φ is a continuous map for every $s > s_0$;
- Φ is proper, which means that for each convergent sequences $(q_n)_{n\in\mathbb{N}}$ and $(\Phi_{g_n}(q_n))_{n\in\mathbb{N}}$ in Q^s, there exists a convergent subsequence $(g_{n_k})_{k\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$ in G;
- each homeomorphism Φ_g commutes with all the maps $j_{(r,s)}$ which means that the action does not depend on s.

As a consequence of the properness assumption, we obtain that the isotropy groups $G_q := \{g \in G \mid \Phi_g(q) = q\}$ are compact.

Finally we suppose that:

(1) For all $s > s_0$ and for all $g \in G$:

$$\Phi_g : Q^s \longrightarrow Q^s \text{ is smooth.} \tag{2.1}$$

(2) For all $s > s_0 + 1$ and for all $q \in Q^s$:

$$\Phi^q : G \longrightarrow Q^{s-1} \text{ is } C^1. \tag{2.2}$$

As a consequence of (2), we obtain that for $s > s_0 + 1$ the infinitesimal generators of this action are not vector fields on Q^s but they are sections of the smooth vector bundle $TQ^{s-1}|Q^s \to Q^s$. Locally, this vector bundle is the product of an open set in the model of Q^s and of the model Banach space of Q^{s-1}. Indeed, denoting by ξ_{Q^s} the infinitesimal generator associated to $\xi \in \mathfrak{g} := T_eG$, we have, for all $q \in Q^s, s > s_0 + 1$:

$$\xi_{Q^s}(q) = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \Phi_{\exp(t\xi)}(q) = T_e\Phi^q(\xi) \in T_qQ^{s-1}.$$

Example A typical example of such an action is

$$\Phi\colon G \times H^s(G,M) \longrightarrow H^s(G,M), \quad \Phi(g,\eta) := \eta \circ R_g$$

where G is a compact finite dimensional Lie group, M is a compact finite dimensional manifold, $s > \frac{\dim G}{2}$, and R_g is the right multiplication in G, that is, $R_g(h) = hg$. Using the fact that G can be viewed as a submanifold of $\mathcal{D}^r(G)$, the group of class H^r-diffeomorphisms of G, $r > \frac{\dim G}{2} + 1$ (see Lemma 2.1 below), and the fact that the composition

$$\circ\colon \mathcal{D}^r(G) \times H^s(G,M) \longrightarrow H^s(G,M), \quad (\gamma,\eta) \longmapsto \eta \circ \gamma$$

is continuous for $r \geq s$ (by Lemma 3.1 of [5]), we obtain that Φ is a continuous action.

For all $g \in G$, the map

$$\Phi_g\colon H^s(G,M) \longrightarrow H^s(G,M), \quad \Phi_g(\eta) = \eta \circ R_g$$

is smooth by the α-Lemma (see Proposition 3.4 of [5]) and the tangent map is given by $T\Phi_g(v_\eta) = v_\eta \circ R_g$, for $v_\eta \in T_\eta H^s(G,M)$. So hypothesis (2.1) is verified.

We now check hypothesis (2.2). For all $\eta \in H^s(G,M), s > \frac{\dim G}{2} + 1$, the map

$$\mathcal{D}^r(G) \longrightarrow H^{s-1}(G,M), \quad \gamma \longmapsto \eta \circ \gamma$$

is of class C^1 by the proof of Proposition 3.4 of [5], for r sufficiently large. Since G is a submanifold of $\mathcal{D}^r(G)$, the map

$$\Phi^\eta\colon G \longrightarrow H^{s-1}(G,M), \quad \Phi^\eta(g) = \eta \circ R_g$$

is of class C^1, and its tangent map is given by $T\Phi^\eta(\xi_g)(h) = T\eta(TL_h(\xi_g))$, for $\xi_g \in T_gG$, and where $L_g(h) = gh$. Remark that $T\Phi^\eta(\xi_g)$ is in $T_{\Phi^\eta(g)}H^{s-1}(G,M)$ and does not belong to $T_{\Phi^\eta(g)}H^s(G,M)$, in general. So the map

$$\Phi^\eta\colon G \longrightarrow H^s(G,M)$$

cannot be C^1 in general.

In Sect. 6 we will prove that with $Q^s = \mathfrak{X}^s_{div}(M), s > s_0 = \frac{\dim M}{2}$, the action

$$l : Iso^+ \times \mathfrak{X}^s_{div}(M) \longrightarrow \mathfrak{X}^s_{div}(M),$$

defined in Sect. 1, is continuous and verifies the hypotheses (2.1) and (2.2).

In a similar way, for the second example treated in the Sect. 1, choosing $Q^s = P^s/SO(2)$, $s > s_0 = \frac{3}{2}$, we see that the action of S^1 induced by R is continuous and verifies the same hypotheses.

We now prove the following Lemma.

Lemma 2.1 *G is a submanifold of $\mathcal{D}^r(G)$, for all $r > \frac{\dim G}{2} + 1$.*

Proof We identify G with the subgroup $\{R_g : G \longrightarrow G \mid g \in G\}$ of $\mathcal{D}^r(G)$. Recall that the model of $\mathcal{D}^r(G)$ is the Hilbert space $T_e\mathcal{D}^r(G) = \mathfrak{X}^r(G)$, consisting of the class H^r vector fields on G. A chart of $\mathcal{D}^r(G)$ at $e = id_G$ is given by

$$\psi : U \subset \mathfrak{X}^r(G) \longrightarrow \mathcal{D}^r(G), \quad \psi(X) = \operatorname{Exp} \circ X,$$

where $\operatorname{Exp} : TG \longrightarrow G$ is the map defined by $\operatorname{Exp} | T_gG = \exp_g$, for $\exp_g(\xi_g) := L_g(\exp(TL_{g^{-1}}(\xi_g)))$, and $\exp : \mathfrak{g} \to G$ the exponential map of the Lie group G.

Let $\mathfrak{X}_L(G) := \{X \in \mathfrak{X}(G) \mid (L_g)^*X = X, \forall g \in G\}$ be the space of left-invariant vector fields on G. Then we have the isomorphism $\mathfrak{g} \longrightarrow \mathfrak{X}_L(G), \quad \xi \longmapsto X_\xi^L$ where $X_\xi^L(g) := T_eL_g(\xi)$. Since $\mathfrak{X}_L(G)$ is finite dimensional, it is a closed subspace of the Hilbert space $\mathfrak{X}^r(G)$ and the topology induced by $\mathfrak{X}^r(G)$ on $\mathfrak{X}_L(G)$ coincides with the one induced by the identification with $\mathfrak{g}$. To show that $\mathfrak{X}_L(G)$ is in fact the model of G viewed as a submanifold of $\mathcal{D}^r(G)$, it suffices to see that $\psi(U \cap \mathfrak{X}_L(G)) = \psi(U) \cap G$. □

Denote by $\operatorname{Orb}(q) := \{\Phi_g(q) \mid g \in G\} \subset Q^s$ the orbit of $q \in Q^s$. Since $\xi_{Q^s}(q)$ is not in T_qQ^s, $\operatorname{Orb}(q)$ can not be a submanifold of Q^s. However we have the following result.

Theorem 2.2 *For all $q \in Q^s, s > s_0 + 1$, $\operatorname{Orb}(q)$ is a submanifold of Q^{s-1} and $T_q \operatorname{Orb}(q) = \{\xi_{Q^s}(q) \mid \xi \in \mathfrak{g}\} \subset T_qQ^{s-1}$.*

Proof The C^1 map $\Phi^q : G \longrightarrow Q^{s-1}$ induces the C^1 map $\widetilde{\Phi}^q : G/G_q \longrightarrow Q^{s-1}$ defined by $\Phi^q = \widetilde{\Phi}^q \circ \pi_{G,G_q}$, where $\pi_{G,G_q} : G \longrightarrow G/G_q$ is the projection given by $\pi_{G,G_q}(g) = gG_q$. Since $\widetilde{\Phi}^q(G/G_q) = \operatorname{Orb}(q)$, we will prove that $\widetilde{\Phi}^q$ is an embedding. It suffices to show that $\widetilde{\Phi}^q$ is a closed injective immersion. This can be proven like in the usual case of a smooth action on a finite dimensional manifold (see Corollary 4.1.22 in Ref. [1] for example). It just remains to show that the range of $T_{[e]}\tilde{\Phi}^q$ is closed and split in T_qQ^{s-1}. This is true because $T_{[e]}\widetilde{\Phi}^q(T_{[e]}(G/G_q))$ is a finite dimensional vector space. □

3 The slice theorem and its consequences

Recall that in the case of a smooth and proper action $\Phi : G \times M \longrightarrow M$ of a finite dimensional Lie group G on a finite dimensional manifold M, the existence of slices is a key fact in the study of the geometric properties of the submanifolds M^H, M_H, and $M_{(H)}$ and of the stratification of the orbit space M/G. In the more general case of a continuous action on an infinite dimensional Banach manifold, we adopt the following definition of a slice (see for example Theorem 7.1 in Ref. [5] or Theorem 4.1 in Ref. [8]).

Definition 3.1 A *slice* at $q \in Q^s$ is a submanifold $S_q \subset Q^s$ containing q such that:

(S1) if $g \in G_q$, then $\Phi_g(S_q) = S_q$,
(S2) if $g \in G$ and $\Phi_g(S_q) \cap S_q \neq \varnothing$, then $g \in G_q$,
(S3) there is a local section $\chi : G/G_q \longrightarrow G$ defined in a neighborhood $V([e])$ of $[e]$ in G/G_q such that the map

$$F : V([e]) \times S_q \longrightarrow Q^s, \quad F([g], q) := \Phi_{\chi([g])^{-1}}(q)$$

is an homeomorphism onto a neighborhood U of q.

Note that usually we take $F([g],q) := \Phi_{\chi([g])}(q)$ but it will be more natural to use $F([g],q) := \Phi_{\chi([g])^{-1}}(q)$.

In finite dimensions and in the case of a smooth action, the previous definition is equivalent to the usual ones (see [14] for equivalent definitions of a slice). We will show that slices exist for a non-smooth action Φ verifying all the hypotheses in Sect. 2.

Note that in Palais [15] it is proven that slices exist for each continuous and proper (in the sense given there) action $G \times X \longrightarrow X$ of a finite dimensional Lie group G on a completely regular topological space X; nevertheless, the construction is not explicit and so it can not be used below to construct charts for Q^s/G.

Recall that the action we consider is proper, so the isotropy groups G_q are compact. Now we prove that for each $q \in Q^s$, there exist a G_q-invariant chart of Q^s at q. This result is shown, for example, in the Appendix B of Cushman and Bates [4], in the case of a smooth and proper action on a finite dimensional manifold. We give below the proof of this statement in order to show carefully that it is still valid in our case of a non-smooth action on an infinite dimensional Banach manifold.

Lemma 3.2 *Assume that the action $\Phi : G \times Q^s \longrightarrow Q^s$ verifies all the hypotheses in Sect. 2. Then for all $q \in Q^s$ there exists a chart (ψ, V) of Q^s at q such that:*

(1) *V is G_q-invariant, and $\psi : V \subset Q^s \longrightarrow T_qQ^s$ verifies $\psi(q) = 0$ and $T_q\psi = \mathrm{id}_{T_qQ^s}$,*
(2) *$\forall r \in V, \forall h \in G_q$, we have:*

$$\psi(\Phi_h(r)) = T_q\Phi_h(\psi(r)).$$

Proof Let (φ, U) be a chart of Q^s at q such that $\varphi(q) = 0_q$ and $T_q\varphi = \mathrm{id}_{T_qQ^s}$. Since G_q is compact, there exists a G_q-invariant neighborhood $V \subset U$ of q; see for example Lemma 2.3.29 of Ortega and Ratiu [14] which does not use in its proof the finite dimensionality of the manifold. Let $W := \varphi(V)$ and let $\overline{\Phi}$ be the action of G_q induced on W by φ, that is, we have the commuting diagram

$$\begin{array}{ccc} \Phi: G_q \times V \subset Q^s & \longrightarrow & V \\ \downarrow \varphi & & \downarrow \varphi \\ \overline{\Phi}: G_q \times W \subset T_qQ^s & \longrightarrow & W. \end{array}$$

We consider the map $\overline{\psi}: W \subset T_qQ^s \longrightarrow T_qQ^s$ defined by

$$\overline{\psi}(v_q) := \int_{G_q} D\overline{\Phi}_{g^{-1}}(0)(\overline{\Phi}_g(v_q))\mathrm{d}g$$

where $D\overline{\Phi}_{g^{-1}}$ is the Fréchet derivative of the smooth map $\overline{\Phi}_{g^{-1}} : W \subset T_qQ^s \longrightarrow T_qQ^s$ and dg is the Haar measure of the compact group G_q such that $\mathrm{Vol}(G_q) = 1$.

With this definition of $\overline{\psi}$ we obtain that (see p. 301 of Cushman and Bates [4] for details):

$$D\overline{\Phi}_h(0)(\overline{\psi}(v_q)) = \overline{\psi}(\overline{\Phi}_h(v_q)), \forall h \in G_q, \forall v_q \in W \quad \text{and} \quad D\overline{\psi}(0) = \mathrm{id}_{T_qQ^s}.$$

Define $\psi := \overline{\psi} \circ \varphi \colon V \longrightarrow T_q Q^s$. Then $T_q\psi = \mathrm{id}_{T_qQ^s}$ and we have

$$\begin{aligned}
&D\overline{\Phi}_h(0)(\overline{\psi}(v_q)) = \overline{\psi}(\overline{\Phi}(v_q))\\
&\Longrightarrow D(\varphi \circ \Phi_h \circ \varphi^{-1})(0)(\psi(\varphi^{-1}(v_q))) = \psi(\Phi_h(\varphi^{-1}(v_q)))\\
&\Longrightarrow T_q\varphi(T_q\Phi_h(T_0\varphi^{-1}(\psi(\varphi^{-1}(v_q))))) = \psi(\Phi_h(\varphi^{-1}(v_q)))\\
&\Longrightarrow T_q\Phi_h(\psi(\varphi^{-1}(v_q))) = \psi(\Phi_h(\varphi^{-1}(v_q))), \text{ since } T_q\varphi = \mathrm{id}_{T_qQ^s}\\
&\Longrightarrow T_q\Phi_h(\psi(r)) = \psi(\Phi_h(r)), \forall\, r \in V.
\end{aligned}$$

□

We will define below a map $\mathcal{S}_q$ that will play a central role in the rest of the paper and is a generalization of the map S defined in Chapter 7 of Karcher [9] in the case of the natural S^1-action on the Hilbert manifold $H^1(S^1, M)$ of closed H^1-curves in a compact Riemannian manifold M without boundary. In order to define this map $\mathcal{S}_q$ we will need the following Lemma.

Lemma 3.3 *Let $\Phi: G \times Q^s \longrightarrow Q^s$ be a continuous action. Let $q \in Q^s$ be such that $H := G_q$ is compact and let U_1 be a neighborhood of q in Q^s. Then there exist a neighborhood $V(H)$ of H in G and a neighborhood U_2 of q in Q^s such that:*

$$\Phi(V(H) \times U_2) \subset U_1.$$

Proof We have $\Phi(h, q) = q, \forall\, h \in H$, so for all $h \in H$, there exist a neighborhood V_h of h in H and a neighborhood U_2^h of q in Q^s such that

$$\Phi(V_h \times U_2^h) \subset U_1.$$

Since $(V_h)_{h\in H}$ is an open covering of the compact group H, there exists a finite subcovering $(V_{h_i})_{i=1\ldots n}$. Let

$$V(H) := \bigcup_{i=1}^{n} V_{h_i} \quad \text{and} \quad U_2 := \bigcap_{i=1}^{n} U_2^{h_i}.$$

Clearly $V(H)$ and U_2 are neighborhoods of H and q; moreover:

$$\Phi(V(H) \times U_2) = \Phi\Big(\bigcup_{i=1}^{n} V_{h_i} \times U_2\Big) \subset \bigcup_{i=1}^{n} \Phi(V_{h_i} \times U_2) \subset \bigcup_{i=1}^{n} \Phi(V_{h_i} \times U_2^{h_i}) \subset U_1.$$

□

Recall that G/H is a smooth manifold (not a group in general) whose elements will be denoted by gH or $[g]$; $\pi_{G,H}\colon G \longrightarrow G/H$ is the projection map.

Assume now that the action $\Phi\colon G \times Q^s \longrightarrow Q^s$ verifies all the hypotheses in Sect. 2. Let $q \in Q^s, s > s_0 + 1$, $H := G_q$, and $\mathfrak{h} := T_eH$. Let $\mathrm{B} := (e_1, \ldots, e_k, e_{k+1}, \ldots, e_n)$ be a basis of $\mathfrak{g}$ such that $(e_1, \ldots, e_k)$ is a basis of $\mathfrak{h}$. Let (φ^{s-1}, U^{s-1}) be a chart of Q^{s-1} at $q \in Q^s$ such that $\varphi^{s-1}(q) = 0_q$ and $T_q\varphi^{s-1} = \mathrm{id}_{T_qQ^{s-1}}$. Let $\chi : G/H \longrightarrow G$ be a local section defined in a neighborhood of $[e]$. We define the map

$$\mathcal{S}_q \colon V([e]) \times U \subset G/H \times Q^s \longrightarrow \mathfrak{n},\ \mathcal{S}_q(gH, r) := \sum_{i=k+1}^{n} \gamma(q) \left(\varphi^{s-1}(\Phi_{\chi(gH)}(r)), E_i(q)\right) e_i$$

where the neighborhoods $V(H)$ and U are such that $\Phi(V(H) \times U) \subset U^{s-1} \cap Q^s$ (which is possible by the preceding Lemma), $V([e])$ is such that $\chi(V([e])) \subset V(H)$,

$\mathfrak{n}$ is the subspace of $\mathfrak{g}$ generated by $(e_{k+1},\ldots,e_n)$, and $E_i(q) := T_e\Phi^q(e_i) \in T_qQ^{s-1}$ is the infinitesimal generator associated to e_i.

Remark that the map $\mathcal{S}_q$ depends on the basis B and on the local section χ. Note that $\mathcal{S}_q(eH,q) = 0$. Note also that if Φ is free at q, we have $H = \{e\}$ and the map $\mathcal{S}_q$ is given by

$$\mathcal{S}_q\colon V(e) \times U \subset G \times Q^s \longrightarrow \mathfrak{g},\ \ \mathcal{S}_q(g,r) := \sum_{i=1}^{n} \gamma(q)\left(\varphi^{s-1}(\Phi_g(r)), E_i(q)\right) e_i. \tag{3.1}$$

The following Lemma states the most important property of the map $\mathcal{S}_q$.

Lemma 3.4 *The map $\mathcal{S}_q$ is of class C^1. Moreover, if we consider the map $\mathcal{S}_q(\_,q)$: $V([e]) \subset G/H \longrightarrow \mathfrak{n}$, then its tangent map at eH,*

$$T_{eH}[\mathcal{S}_q(\_,q)] : T_{eH}(G/H) \longrightarrow \mathfrak{n}$$

is invertible. More precisely, its matrix relative to the basis $(e_{k+1},\ldots,e_n)$ is given by

$$\left(T_{eH}[\mathcal{S}_q(\_,q)]\right) = \left(\gamma(q)(E_i(q),E_j(q))\right)_{i,j=k+1,\ldots,n}$$

Proof By the assumptions (2.1) and (2.2) on the action, we obtain that the map

$$V([e]) \times U \longrightarrow U^{s-1},\ \ (gH,r) \longmapsto \Phi_{\chi(gH)}(r)$$

is C^1 as a map with values in Q^{s-1}. Using that φ^{s-1} is a smooth chart for Q^{s-1} and that the bilinear form $\gamma(q)$ is continuous, we obtain that $\mathcal{S}_q$ is C^1. Now we compute the tangent map to $\mathcal{S}_q(\_,q)\colon V([e]) \subset G/H \longrightarrow \mathfrak{n}$ at eH. Consider the tangent map $T_e\pi_{G,H} : \mathfrak{g} \longrightarrow T_{eH}(G/H)$; clearly we have $\ker(T_e\pi_{G,H}) = \mathfrak{h}$ and we obtain that $T_e\pi_{G,H}\colon \mathfrak{n} \longrightarrow T_{eH}(G/H)$ is bijective, so $(T_e\pi_{G,H}(e_{k+1}),\ldots,T_e\pi_{G,H}(e_n))$ is a basis of $T_{eH}(G/H)$. Since $\pi_{G,H}\circ\exp(te_j)$ is a curve in G/H tangent to $T\pi_{G,H}(e_j)$ at eH, we have:

$$\begin{aligned}
T_{eH}[\mathcal{S}_q(_,q)](T\pi_{G,H}(e_j)) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{S}_q(\pi_{G,H}(\exp(te_j)),q)\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \sum_{i=k+1}^{n} \gamma(q)\left(\varphi^{s-1}(\Phi_{\chi(\pi_{G,H}(\exp(te_j)))}(q)), E_i(q)\right) e_i\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \sum_{i=k+1}^{n} \gamma(q)\left(\varphi^{s-1}(\Phi_{\exp(te_j)}(q)), E_i(q)\right) e_i\\
&= \sum_{i=k+1}^{n} \gamma(q)\left(T_q\varphi^{s-1}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_{\exp(te_j)}(q)\right), E_i(q)\right) e_i\\
&= \sum_{i=k+1}^{n} \gamma(q)(E_j(q),E_i(q))e_i.
\end{aligned}$$

So we obtain that

$$\left(T_{eH}[\mathcal{S}_q(\_,q)]\right) = \left(\gamma(q)(E_i(q),E_j(q))\right)_{i,j=k+1,\ldots,n}$$

and since $\gamma(q)$ is strongly non-degenerate on any finite dimensional vector space, it remains to show that $(E_{k+1}(q),\ldots,E_n(q))$ is linearly independent. To prove this, we consider the tangent map $T_e\Phi^q : \mathfrak{g} \longrightarrow T_q Q^{s-1}$. We have $\ker(T_e\Phi^q) = \mathfrak{h}$. Indeed, if $\xi \in \mathfrak{g}$ is such that $T_e\Phi^q(\xi) = 0$ then we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\exp(\mathrm{t}\xi)}(q) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Phi_{\exp((\mathrm{t}+\mathrm{s})\xi)}(q) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Phi_{\exp(\mathrm{t}\xi)}(\Phi_{\exp(\mathrm{s}\xi)}(q))$$
$$= T_q\Phi_{\exp(\mathrm{t}\xi)}(T_e\Phi^q(\xi)) = 0,$$

and we conclude that $\Phi_{\exp(\mathrm{t}\xi)}(q) = \Phi_{\exp(0\xi)}(q) = q$ for all t, so $\exp(\mathrm{t}\xi) \in G_q = H$ and $\xi \in \mathfrak{h}$. Thus we obtain that $T_e\Phi^q : \mathfrak{n} \longrightarrow T_q Q^{s-1}$ is injective, and $(E_{k+1}(q), \ldots, E_n(q))$ is linearly independent since it is the image of the basis $(e_{k+1}, \ldots, e_n)$ of $\mathfrak{n}$ under the injective linear map $T_e\Phi^q$. □

For $q \in Q^s, s > s_0 + 1$, we define the closed codimension $(n-k)$ subspace

$$N_q^s := \{v_q \in T_q Q^s \mid \gamma(q)(v_q, E_i(q)) = 0, \quad \text{for all } i = k+1, \ldots, n\} \tag{3.2}$$

in $T_q Q^s$, where as before, $\mathrm{B} = (e_1, \ldots, e_k, e_{k+1}, \ldots, e_n)$ is a basis of $\mathfrak{g}$ such that $(e_1, \ldots, e_k)$ is a basis of $\mathfrak{h} := T_e G_q$, and $E_i(q)$ are the corresponding infinitesimal generators. We can now state the following useful consequence of the properties of $\mathcal{S}_q$.

Theorem 3.5 *There exists a neighborhood $V([e])$ of eH in G/H, a neighborhood U of q in Q^s, and a map $\beta_q : U \longrightarrow V([e])$ of class C^1 such that for all $[g] \in V([e])$ and $r \in U$ we have:*

$$\varphi^{s-1}(\Phi_{\chi([g])}(r)) \in N_q^s \Longleftrightarrow \mathcal{S}_q([g], r) = 0 \Longleftrightarrow [g] = \beta_q(r).$$

Proof The first equivalence follows from the definition of $\mathcal{S}_q$. For the second it suffices to use the implicit function theorem since the map $\mathcal{S}_q$ is of class C^1 and verifies $\mathcal{S}_q(eH, q) = 0$ and $T_{eH}[\mathcal{S}_q(\_, q)]$ is invertible, by the preceding Lemma. □

Theorem 3.6 (Slice theorem) *Assume that the action verifies all the hypotheses in Sect. 2. Then for each $q \in Q^s, s > s_0 + 1$, there exists a slice at q.*

Proof Let $H := G_q$. By Lemma 3.2, there exists a H-invariant chart (ψ, V) of q at Q^s. Let $\overline{\gamma}$ be the metric defined by

$$\overline{\gamma}(q)(u_q, v_q) := \int_H \gamma(\Phi_h(q))(T\Phi_h(u_q), T\Phi_h(v_q))\mathrm{d}h,$$

where $\mathrm{d}h$ is the Haar measure on H such that $\mathrm{Vol}(H) = 1$. Then we have

$$(\Phi_h)^*\overline{\gamma} = \overline{\gamma}, \forall\, h \in H.$$

Consider the closed codimension $(n-k)$ subspace $N_q^s := \{v_q \in T_q Q^s \mid \overline{\gamma}(q)(v_q, E_i(q)) = 0,$ for all $i = k+1, \ldots, n\}$, and define the submanifold

$$\widetilde{S}_q := \psi^{-1}(N_q^s \cap \psi(V)).$$

By invariance of ψ and N_q^s, we obtain that $\Phi_h(\widetilde{S}_q) = \widetilde{S}_q$ for all $h \in H$, so condition (S1) is satisfied.

We now check condition (S3). Let $\chi : W([e]) \subset G/H \longrightarrow G$ be a local section defined in a neighborhood $W([e])$ of $[e]$ and consider the continuous map

$$F : V([e]) \times S_q \longrightarrow Q^s, \;\; F([g], q) := \Phi_{\chi([g])^{-1}}(q).$$

By Theorem 3.5, there is a neighborhood $V([e])$ of eH in G/H, a neighborhood U of q in Q^s, and a C^1 map $\beta_q \colon U \longrightarrow V([e])$ such that

$$\psi(\Phi_{\chi([g])}(r)) \in N_q^s \Longleftrightarrow [g] = \beta_q(r).$$

By definition of $\widetilde{S}_q$ we obtain

$$\Phi_{\chi([g])}(r) \in \widetilde{S}_q \Longleftrightarrow [g] = \beta_q(r),$$

so we can define the continuous map

$$U \subset Q^s \longrightarrow V([e]) \times \widetilde{S}_q, \quad r \longmapsto (\beta_q(r), \Phi_{\chi(\beta_q(r))}(r))$$

and one sees that it is the inverse of the map F. This proves that F is a homeomorphism. Remark that for (S1) and(S3) we do not use the properness of the action.

We now show that there exists a neighborhood U of q such that choosing $S_q := \widetilde{S}_q \cap U$ we have condition (S2):

$$\Phi_g(S_q) \cap S_q \neq \emptyset \Longrightarrow g \in H.$$

By contradiction, suppose that such a neighborhood does not exist. So, without loss of generality, we can consider a H-invariant fundamental system of neighborhoods $(U^m)_{m\in\mathbb{N}}$ of q such that for all $m \in \mathbb{N}$ we have the following property:

there exists $g_m \notin H$ such that $\Phi_{g_m}(S_q^m) \cap S_q^m \neq \emptyset$, where $S_q^m := \widetilde{S}_q \cap U^m$.

Note that $\Phi_{g_m}(S_q^m) \cap S_q^m \neq \emptyset$ if and only if $\Phi_{g_m^{-1}}(S_q^m) \cap S_q^m \neq \emptyset$. So for all m there exists $g_m \notin H$ and $r_m \in S_q^m$ such that $\Phi_{g_m}(r_m) \in S_q^m$. Then we have $r_m \longrightarrow q$ and $\Phi_{g_m}(r_m) \longrightarrow q$ and by properness of the action we obtain the existence of a subsequence $(g_{m_k})_{k\in\mathbb{N}} \subset (g_m)_{m\in\mathbb{N}}$ and of an element $g \in G$ such that $g_{m_k} \longrightarrow g$. Therefore, $\Phi_g(q) = q$ and hence $g \in H$.

Let $V([e])$ be a neighborhood of $[e]$ in G/H and $V(H) := \pi_{G,H}^{-1}(V([e])$ a neighborhood of H in G. We can suppose that $g_{m_k} \in V(H)$. Since $\Phi_{g_{m_k}}(r_{m_k}) \in S_q^{m_k}$, we have

$$S_q^{m_k} \ni \Phi_{g_{m_k}}(r_{m_k}) = \Phi_{\chi(\pi_{G,H}(g_{m_k}))h}(r_{m_k}) = \Phi_{\chi(\pi_{G,H}(g_{m_k}))}(\Phi_h(r_{m_k})),$$

where $h \in H$. For sufficiently small $V([e])$, we have $[g_{m_k}] = \beta_q(\Phi_h(r_{m_k}))$. But by (S1) we know that $\Phi_h(r_{m_k}) \in S_q^m$ and hence $\beta_q(\Phi_h(r_{m_k})) = eH$. Thus we obtain that $[g_{m_k}] = eH$ and so $g_{m_k} \in H$. This last affirmation is a contradiction. □

Here are two immediate consequences of the Slice Theorem:

(1) For all $s \in F(V([e]) \times S_q) \cap S_q$ we have $G_s \subset G_q$.
Indeed, $g \in G_s$ implies $\Phi_g(s) = s$, but since $s \in S_q$ we obtain $\Phi_g(S_q) \cap S_q \neq \emptyset$. So by (S2) we have $g \in G_q$.

(2) For all $r \in F(V([e]) \times S_q)$, G_r is conjugated to a subgroup of G_q.
Indeed, $r \in F(V([e]) \times S_q)$ implies $r = \Phi_{\chi([g])^{-1}}(s)$ with $s \in S_q$. So G_r is conjugated to G_s. Since $s \in F(V([e]) \times S_q) \cap S_q$, by (1) we obtain that $G_s \subset G_q$.

We now study the geometric properties of some important subsets of Q^s. If H is a subgroup of G, we denote by

$$(Q^s)^H := \{q \in Q^s \mid G_q \supset H\}$$

the *H-fixed point set*, by

$$(Q^s)_H = \{q \in Q^s \mid G_q = H\}$$

the *H-isotropy type set*, and by

$$(Q^s)_{(H)} := \{q \in Q^s \mid G_q \text{ is conjugated to } H\}$$

the (H)*-orbit type set*. We have $(Q^s)_H \subset (Q^s)^H$. It is well known that in the case of a smooth and proper action $\Phi\colon G \times M \longrightarrow M$ on a finite dimensional manifold M, we obtain that M^H, M_H, and $M_{(H)}$ are submanifolds of M and that M_H is an open subset of M^H. We will see that some of these results are still valid in our case.

Define for $q \in (Q^s)^H$ the closed subspace

$$[T_q Q^s]^H := \{v_q \in T_q Q^s \mid T_q \Phi_h(v_q) = v_q, \quad \forall h \in H\}.$$

Theorem 3.7 *Assume that the action $\Phi\colon G \times Q^s \longrightarrow Q^s$ verifies all the hypotheses in Sect. 2. Let H be a compact subgroup of G. Then $(Q^s)^H$ is a smooth submanifold of Q^s whose tangent space at $q \in (Q^s)^H$ is $T_q(Q^s)^H = [T_q Q^s]^H$.*

Proof If $q \in (Q^s)^H$, using the preceding lemma and the fact that $H \subset G_q$, there exists a chart (ψ, V) of Q^s at q such that $\psi(\Phi_h(r)) = T_q\Phi_h(\psi(r)), \forall r \in V, \forall h \in G_q$. So we obtain that $\psi(V \cap (Q^s)^H) = \psi(V) \cap [T_q Q^s]^H$, which proves that $(Q^s)^H$ is a submanifold of Q^s whose tangent space at q is given by $T_q(Q^s)^H = [T_q Q^s]^H$. □

Theorem 3.8 *Assume that the action $\Phi\colon G \times Q^s \longrightarrow Q^s$ verifies all the hypotheses in Sect. 2. Let H be a compact subgroup of G. Then for all $s > s_0 + 1$, $(Q^s)_H$ is an open subset of the manifold $(Q^s)^H$.*

Proof Let $q \in (Q^s)_H$. We will find a neighborhood of q in $(Q^s)^H$ which is included in $(Q^s)_H$. It suffices to consider the open set $(Q^s)^H \cap F(V([e]) \times S_q)$ in $(Q^s)^H$. Indeed for each $r \in (Q^s)^H \cap F(V([e]) \times S_q)$ we have $H \subset G_r$ and by the consequence (2) of the Slice Theorem, we know that G_r is conjugated to a subgroup of $G_q = H$. So we must have $G_r = H$ and we obtain $r \in (Q^s)_H$. This proves that $(Q^s)^H \cap F(V([e]) \times S_q)$ is an open neighborhood of q in $(Q^s)_H$. □

Corollary 3.9 *Assume that the action $\Phi\colon G \times Q^s \longrightarrow Q^s$ verifies all the hypotheses in Sect. 2. Let H be a compact subgroup of G. Then for $s > s_0+1$, $(Q^s)_H$ is a submanifold of Q^s.*

We have not succeeded in proving that the (H)-orbit type set $(Q^s)_{(H)}$ is a submanifold of Q^s. In fact, all the proofs of this statement in the usual finite dimensional situation use the smoothness of Φ on $G \times M$ in a crucial manner. However, we will show in the next section in which sense $(Q^s)_{(H)}$ carries a smooth structure.

4 Geometric properties of the orbit space and of the (H)-orbit type sets

In this section we assume that the action Φ verifies all the hypotheses made in Sect. 2. Moreover we suppose that *all the isotropy groups are conjugated to a given one*, say H. We consider the *orbit space* Q^s/G endowed with the quotient topology. Recall that the projection $\pi : Q^s \longrightarrow Q^s/G$ is a continuous open map. We will denote by $\omega, \upsilon \ldots \in Q^s/G$ the orbits and by $q_\omega, q_\upsilon, \ldots \in Q^s$ the elements of Q^s that belong to the corresponding orbits, that is, $q_\omega \in \pi^{-1}(\omega), q_\upsilon \in \pi^{-1}(\upsilon)$.

The first goal of this section is to show that the orbit space Q^s/G is a (Banach) topological manifold. In order to do that, we will construct explicit charts for Q^s/G. We will see that Q^s/G is not a smooth manifold. However, with these charts, we will show in the second part of this section, that Q^s/G shares many properties with a C^1-manifold. For example, we will be able to define a notion of tangent bundle as well as a notion of differentiable function for Q^s/G. Our construction of charts is inspired by the method used by H. Karcher to prove the local contractibility of the set $H^1(S^1, M)/S^1$ (see Chapter 7 of Karcher [9]). This construction is done in several steps given in the following two lemmas. Recall at this point the definition of N_q^s in (3.2).

Lemma 4.1 *Let $q \in Q^s, s > s_0 + 1$, and let (ψ, V) be a G_q-invariant chart of Q^s at q (whose existence is proved in Lemma 3.2). Then for V sufficiently small, the continuous map*

$$\pi \circ \psi^{-1} : N_q^s \cap \psi(V) \longrightarrow Q^s/G$$

is injective.

Proof For V sufficiently small, $S_q := \psi^{-1}(N_q^s \cap \psi(V))$ is a slice at q. Let $u_q, v_q \in N_q^s \cap \psi(V)$ such that $\pi(\psi^{-1}(u_q)) = \pi(\psi^{-1}(v_q))$. Then we have $\pi(s_1) = \pi(s_2)$ with $s_1 = \psi^{-1}(u_q), s_2 = \psi^{-1}(v_q) \in S_q$. So we have $s_1 = \Phi_g(s_2)$ and we obtain $\Phi_q(S_q) \cap S_q \neq \varnothing$. By (S2), we have $g \in G_q$. By the consequence (1) of the Slice Theorem, we know that $G_{s_2} \subset G_q$, but since all the isotropy groups are conjugated, we have $G_{s_2} = G_q$. So we obtain that $s_1 = \Phi_g(s_2) = s_2$. This proves that $u_q = v_q$. □

Lemma 4.2 *Let $q \in Q^s, s > s_0 + 1$, and let (ψ, V) be a chart of Q^s at q. We can choose a neighborhood U of q in Q^s such that the continuous map*

$$B_q : U \subset Q^s \longrightarrow N_q^s, \quad B_q(r) := \psi(\Phi_{\chi(\beta_q(r))}(r))$$

does not depend on the choice of the representative in the equivalence class of r intersected with U. So, for all $\omega \in Q^s/G$ and $q_\omega \in \pi^{-1}(\omega)$ we can define the map $\mathcal{B}_{q_\omega}$ such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[d, "\pi"'] \arrow[dr, "B_{q_\omega}"] & \\ \mathcal{U} \arrow[r, "\mathcal{B}_{q_\omega}"'] & N^s_{q_\omega}, \end{tikzcd}$$

where $\mathcal{U} := \pi(U)$ is a neighborhood of ω in Q^s/G. Furthermore, the map $\mathcal{B}_{q_\omega}$ is continuous and injective.

Proof Let $r, \bar{r} \in U$ be in the same orbit, that is, $\bar{r} = \Phi_g(r)$. We show that $B_{q_\omega}(r) = B_{q_\omega}(\bar{r})$. We have

$$\Phi_{\chi(\beta_q(\bar{r}))}(\bar{r}) = \Phi_{\chi(\beta_q(\bar{r}))g}(r) = \Phi_{\chi(\beta_q(\bar{r}))g\chi(\beta_q(r))^{-1}}(\Phi_{\chi(\beta_q(r))}(r)) = \Phi_h(\Phi_{\chi(\beta_q(r))}(r)),$$

where $h := \chi(\beta_q(\bar{r}))g\chi(\beta_q(r))^{-1})$. Since $\Phi_{\chi(\beta_q(\bar{r}))}(\bar{r}), \Phi_{\chi(\beta_q(r))}(r) \in S_q$, we have $h \in G_q$ by (S2). So we obtain that $\Phi_h(\Phi_{\chi(\beta_q(r))}(r)) = \Phi_{\chi(\beta_q(r))}(r)$, by the consequence (1) of the Slice Theorem combined with the fact that all isotropy groups are conjugated.

Since the value $B_{q_\omega}(r)$ does not depend on the choice of the representative in the equivalence class of r intersected with U, we can define the map $\mathcal{B}_{q_\omega} : \mathcal{U} \longrightarrow N^s_{q_\omega}$ such that $B_{q_\omega} = \mathcal{B}_{q_\omega} \circ \pi$; it is continuous by construction.

We now show that $\mathcal{B}_{q_\omega}$ is injective. So let $\upsilon, \tau \in \mathcal{U}$ such that $\mathcal{B}_{q_\omega}(\upsilon) = \mathcal{B}_{q_\omega}(\tau)$. Then choosing any $r_\upsilon \in \pi^{-1}(\upsilon) \cap U$ and $r_\tau \in \pi^{-1}(\tau) \cap U$ we have:

$$\begin{aligned}\mathcal{B}_{q_\omega}(\upsilon) = \mathcal{B}_{q_\omega}(\tau) &\Rightarrow B_{q_\omega}(r_\upsilon) = B_{q_\omega}(r_\tau)\\ &\Rightarrow \Phi_{\chi(\beta_{q_\omega}(r_\upsilon))}(r_\upsilon) = \Phi_{\chi(\beta_{q_\omega}(r_\tau))}(r_\tau)\\ &\Rightarrow \pi(r_\upsilon) = \pi(r_\tau)\\ &\Rightarrow \upsilon = \tau. \end{aligned}$$

□

Theorem 4.3 *Let $\omega \in Q^s/G, s > s_0 + 1$, $q_\omega \in \pi^{-1}(\omega)$, and (ψ, V) a chart of Q^s at q_ω. We can choose the neighborhood $\mathcal{U}$ of ω in Q^s/G such that the map*

$$\mathcal{B}_{q_\omega} : \mathcal{U} \subset Q^s/G \longrightarrow \mathcal{B}_{q_\omega}(\mathcal{U}) \subset N^s_{q_\omega}$$

is an homeomorphism between open subsets with inverse given by $\pi \circ \psi^{-1}$. So Q^s/G is a topological manifold (with the quotient topology) modeled on the Banach space $N^s_{q_\omega}$.

Proof We proceed in several steps.

(1) By Lemma 4.1, for V sufficiently small, the continuous map $\pi \circ \psi^{-1} : N^s_{q_\omega} \cap \psi(V) \longrightarrow Q^s/G$ is injective.

(2) By Lemma 4.2 there is a neighborhood $\mathcal{U}$ of ω such that $\mathcal{B}_{q_\omega} : \mathcal{U} \longrightarrow \mathcal{B}_{q_\omega}(\mathcal{U})$ is continuous and bijective. We can choose $\mathcal{U}$ such that $\mathcal{B}_{q_\omega}(\mathcal{U}) \subset N^s_{q_\omega} \cap \psi(V)$.

(3) So we can compose the maps

$$\mathcal{U} \xrightarrow{\ \mathcal{B}_{q_\omega}\ } \mathcal{B}_{q_\omega}(\mathcal{U}) \xrightarrow{\ \pi\circ\psi^{-1}\ } \pi\left(\psi^{-1}\left(\mathcal{B}_{q_\omega}(\mathcal{U})\right)\right)$$

to get

$$\left(\pi \circ \psi^{-1} \circ \mathcal{B}_{q_\omega}\right)(\upsilon) = \pi\left(\psi^{-1}\left(B_{q_\omega}(r_\upsilon)\right)\right) = \pi\left(\psi^{-1}\left(\psi\left(\Phi_{\chi(\beta_{q_\omega}(r_\upsilon))}(r_\upsilon)\right)\right)\right) = \upsilon,$$

that is, $\pi \circ \psi^{-1} \circ \mathcal{B}_{q_\omega}$ is the identity map on $\mathcal{U}$. Thus we conclude that:

- $\pi\left(\psi^{-1}\left(\mathcal{B}_{q_\omega}(\mathcal{U})\right)\right) = \mathcal{U}$, so it is an open subset of Q^s/G,
- $\mathcal{B}_{q_\omega}(\mathcal{U}) = (\pi \circ \psi^{-1})^{-1}(\mathcal{U})$, so it is an open subset of $N^s_{q_\omega}$, and
- $\pi \circ \psi^{-1} = \left(\mathcal{B}_{q_\omega}\right)^{-1}$.

This shows that Q^s/G is a topological manifold (with the quotient topology) modeled on the Banach space $N^s_{q_\omega}$. □

Now a natural question arises: is Q^s/G a differentiable manifold relative to the naturally induced differentiable structure? The answer is no. By contradiction, suppose it is the case, so $\pi : Q^s \longrightarrow Q^s/G$ is a locally trivial fiber bundle and we can take the tangent map $T_q\pi : T_qQ^s \longrightarrow T_{\pi(q)}(Q^s/G)$ whose kernel is the vertical subspace $V_qQ^s = \ker(T_q\pi)$. We know that the vertical subspace is generated by the infinitesimal generators at q, that is, $V_qQ^s = \{\xi_{Q^s}(q) \mid \xi \in \mathfrak{g}\}$. This is a contradiction because, in general, $\xi_{Q^s}(q) \notin T_qQ^s$ (recall that we only have $\xi_{Q^s}(q) \in T_qQ^{s-1}$). Since Q^s/G

is not a differentiable manifold, we cannot define the tangent bundle or C^1 curves in the usual way. To overcome this difficulty, we will use that changes of charts (which are not differentiable in the usual sense) are not only homeomorphisms, but are also differentiable relative to a weaker topology (in fact H^{s-1}). This agrees with the fact that our vertical subspace is a subset of $TQ^{s-1}|Q^s$.

For simplicity we suppose that the action Φ is free, but all the following results are still true in the case when all the isotropy groups are conjugated.

In the following definition, we give a generalization of the notion of C^1 curves in Q^s and N_q^s. Differentiating these curves we obtain a generalization of the notion of tangent vectors to Q^s and N_q^s.

Definition 4.4 Let $s > s_0 + 1$.

(i) For I open in $\mathbb{R}$ we define the set

$$C_W^1(I, Q^s) := C^0(I, Q^s) \cap C^1(I, Q^{s-1})$$

of continuous curves in Q^s which are continuously differentiable with respect to the Q^{s-1} differentiable structure. The set $C_W^1(I, N_q^s)$ is defined in the same way. Such curves will be called *weakly-differentiable*.

(ii) For $q \in Q^s$, the *weak tangent space* of Q^s at q is defined by

$$T_q^W Q^s := \{\dot{d}(0) \in T_q Q^{s-1} \mid d \in C_W^1(I, Q^s), d(0) = q\}.$$

The set $T_{v_q}^W N_q^s$ is defined in the same way.

(iii) The *weak tangent bundle* of Q^s is defined by

$$T^W Q^s := \bigcup_{q \in Q^s} T_q^W Q^s.$$

The set $T^W N_q^s$ is defined in the same way.

Note that $T_q^W Q^s$ is a vector space and that we have the inclusions $T_q Q^s \subset T_q^W Q^s \subset T_q Q^{s-1}$, for all $q \in Q^s$. The same is true for N_q^s. Since N_q^s is a vector space, $T_{v_q}^W N_q^s$ does not depend on v_q and we prefer the notation $(N_q^s)^W := T_{v_q}^W N_q^s$. The previous inclusions become in this case $N_q^s \subset (N_q^s)^W \subset N_q^{s-1}$ and the tangent bundle is $T^W N_q^s = N_q^s \times (N_q^s)^W$. We endow $T_q^W Q^s$ and $(N_q^s)^W$ with the topology of $T_q Q^{s-1}$ and N_q^{s-1}, respectively. Note that for all $q \in Q^s$ and $\xi \in \mathfrak{g}$, we have $\xi_{Q^s}(q) \in T_q^W Q^s$, since $\xi_{Q^s}(q) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_{\exp(t\xi)}(q)$ and the curve $t \mapsto \Phi_{\exp(t\xi)}(q)$ is in $C_W^1(I, Q^s)$ by the hypothesis 2.2 on the G-action.

Now we consider the regularity of the map B_q. To be more precise we will use the notation B_q^s instead of B_q. Recall that B_q^s is given by

$$B_q^s \colon U \subset Q^s \longrightarrow N_q^s, \;\; B_q^s(r) := \varphi^s(\Phi_{\beta_q(r)}(r)).$$

Note that β_q is C^1 by Theorem 3.5 and that Φ is C^1 as a map with values in Q^{s-1} by the working hypothesis on the action (2.2). Hence $r \in U \mapsto \Phi_{\beta_q(r)}(r) \in Q^{s-1}$ is a C^1 map. Since we can think of the chart φ^s of Q^s at q as the restriction of a chart φ^{s-1} of Q^{s-1} at q, we can choose U such that

$$B_q^s \in C^1\left(U, N_q^{s-1}\right).$$

Interpreted this way, we can compute the tangent map at r in the direction $v_r \in TQ^s|U$. We have by Lemma 4.2

$$T_r B_q^s(v_r) = T_{\Phi_{\beta_q(r)}(r)}\varphi^{s-1}\left(T_r\Phi_{\beta_q(r)}(v_r) + T_{\beta_q(r)}\Phi^r(T_r\beta_q(v_r))\right). \tag{4.1}$$

Now we will show that this tangent map

$$TB_q^s\colon TQ^s|U \longrightarrow N_q^s \times N_q^{s-1}$$

makes sense and is continuous on the larger space $TQ^{s-1}|U$. Indeed, by the assumption (2.1), the term $T_r\Phi_{\beta_q(r)}(v_r)$ makes sense for all $v_r \in TQ^{s-1}|U$. It remains to show that $T_r\beta_q(v_r)$ makes sense for $v_r \in TQ^{s-1}|U$.

By Theorem 3.5 we have $\mathcal{S}_q(\beta_q(r), r) = 0$ for r in a neighborhood of q. Differentiating this expression relative to r, we obtain

$$T_{\beta_q(r)}[\mathcal{S}_q(\_, r)](T_r\beta_q(v_r)) + T_r[\mathcal{S}_q(\beta_q(r), \_)](v_r) = 0.$$

By Theorem 3.5 again, we know that $T_{\beta_q(r)}[\mathcal{S}_q(\_, r)]$ is invertible in a neighborhood of q, so we obtain

$$T_r\beta_q(v_r) = -\left(T_{\beta_q(r)}[\mathcal{S}_q(\_, r)]\right)^{-1} T_r[\mathcal{S}_q(\beta_q(r), \_)](v_r).$$

Thus it suffices to show that $T_r[\mathcal{S}_q(\beta_q(r), \_)](v_r)$ is well-defined for $v_r \in TQ^{s-1}|U$. This is true because (3.1) implies the formula

$$T_r[\mathcal{S}_q(g, \_)](v_r) = \sum_{i=1}^{n} \gamma(q)\left(T_{\Phi_g(r)}\varphi^{s-1}(T\Phi_g(v_r)), E_i(q)\right) e_i.$$

So the tangent map TB_q^s makes sense on $TQ^{s-1}|U$ and therefore on $T^W Q^s|U$. We will denote by

$$T^W B_q^s\colon T^W Q^s|U \longrightarrow N_q^s \times N_q^{s-1}$$

this extension. Remark that if $s > s_0+2$ we have $T^W B_q^s = TB_q^{s-1}$ on $T^W Q^s|U$. The following Lemma shows that $T^W B_q^s$ takes values in $T^W N_q^s = N_q^s \times (N_q^s)^W \subset N_q^s \times N_q^{s-1}$, that is,

$$T^W B_q^s\colon T^W Q^s|U \longrightarrow T^W N_q^s = N_q^s \times (N_q^s)^W.$$

Lemma 4.5 *Let $q \in Q^s, s > s_0 + 2$, and U neighborhood of q in Q^s such that B_q^s is defined on U. Let $d \in C_W^1(I, Q^s)$ such that $d(t) \in U$ for all $t \in I$. Then $B_q^s \circ d \in C_W^1(I, N_q^s)$ and we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}(B_q^s \circ d)(t) = T_{d(t)}^W B_q^s(\dot{d}(t)). \tag{4.2}$$

Proof It is clear that $B_q^s \circ d \in C^0(I, N_q^s)$ so we have to show that $B_q^s \circ d \in C^1(I, N_q^{s-1})$. Recall that $B_q(d(t)) = \varphi^s(\Phi_{\beta_q(d(t))}(d(t)))$. For all $u \in I$, by (2.1), we have

$$\begin{array}{ccccc} I & \xrightarrow{C^1} & Q^{s-1} & \xrightarrow{C^\infty} & Q^{s-1} \\ t & \longmapsto & d(t) & \longmapsto & \Phi_{\beta_q(d(u))}(d(t)). \end{array}$$

Since $d \in C^1(I, Q^{s-1})$, using Theorem 3.5 (for $s-1$, which is allowed since we assume $s > s_0 + 2$), and (2.2), we get for all $t \in I$

$$\begin{array}{ccccccc} I & \stackrel{C^1}{\longrightarrow} & Q^{s-1} & \stackrel{C^1}{\longrightarrow} & G & \stackrel{C^1}{\longrightarrow} & Q^{s-1} \\ u & \longmapsto & d(u) & \longmapsto & \beta_q(d(u)) & \longmapsto & \Phi_{\beta_q(d(u))}(d(t)) \end{array}$$

So $B_q^s \circ d \in C^1(I, N_q^{s-1})$.

To prove formula (4.2) recall that $d \in C^1(I, Q^{s-1})$ and $B_q^{s-1} \in C^1(U, N_q^{s-2})$, so by the chain rule we have $B_q^{s-1} \circ d \in C^1(I, N_q^{s-2})$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(B_q^{s-1} \circ d)(t) = T_{d(t)} B_q^{s-1}(\dot{d}(t)).$$

Since $d(t) \in U$ we have $B_q^{s-1} \circ d = B_q^s \circ d$, and as was discussed before, $T_{d(t)} B_q^{s-1}(\dot{d}(t)) = T_{d(t)}^W B_q^s(\dot{d}(t))$. So we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}(B_q^s \circ d)(t) = T^W B_q^s(\dot{d}(t))$$

where $B_q^s \circ d$ is derived as a curve in $C^1(I, N_q^{s-2})$. But we know that we have in fact $B_q^s \circ d \in C^1(I, N_q^{s-1})$ so in the preceding formula $B_q^s \circ d$ is derived as a curve in $C^1(I, N_q^{s-1})$. □

Now we state an important result of this section. It says in which sense the change of charts is differentiable. This will be useful in the construction of a tangent bundle of Q^s/G.

Theorem 4.6 *Let $\omega, \upsilon \in Q^s/G, s > s_0 + 2$, and $(\mathcal{U}_1, \mathcal{B}_{q_\omega}), (\mathcal{U}_2, \mathcal{B}_{q_\upsilon})$ be two charts of Q^s/G such that $\mathcal{U}_1 \cap \mathcal{U}_2 \neq \varnothing$. Then the change of charts*

$$F := \mathcal{B}_{q_\upsilon} \circ \mathcal{B}_{q_\omega}^{-1} : \mathcal{B}_{q_\omega}(\mathcal{U}_1 \cap \mathcal{U}_2) \subset N_{q_\omega}^s \longrightarrow \mathcal{B}_{q_\upsilon}(\mathcal{U}_1 \cap \mathcal{U}_2) \subset N_{q_\upsilon}^s$$

is C^1 as a map with values in $N_{q_\upsilon}^{s-1}$.

Proof By construction of the charts, see Lemma 4.2, we have $\mathcal{U}_1 = \pi(U_1)$ and $\mathcal{U}_1 = \pi(U_1)$ where U_1 is a neighborhood of q_ω and U_2 a neighborhood of q_υ. Recall that $\mathcal{B}_{q_\upsilon}$ is defined such that $\mathcal{B}_{q_\upsilon} \circ \pi = B_{q_\upsilon} : U_2 \longrightarrow N_{q_\upsilon}^s$, where $B_{q_\upsilon}(r) = \psi^s(\Phi_{\beta_{q_\upsilon}(r)}(r))$.

Let $v_{q_\omega} \in \mathcal{B}_{q_\omega}(\mathcal{U}_1 \cap \mathcal{U}_2)$ and let $\tau := \mathcal{B}_{q_\omega}^{-1}(v_{q_\omega}) = (\pi \circ (\varphi^s)^{-1})(v_{q_\omega}) \in \mathcal{U}_1 \cap \mathcal{U}_2$. Since $\pi((\varphi^s)^{-1}(v_{q_\omega})) \in \mathcal{U}_2 = \pi(U_2)$, there exists $g_0 \in G$ such that $\Phi_{g_0}((\varphi^s)^{-1}(v_{q_\omega})) \in U_2$. By continuity of Φ_{g_0}, there exists a neighborhood V of $(\varphi^s)^{-1}(v_{q_\omega})$ such that $\Phi_{g_0}(V) \in U_2$. Since $v_{q_\omega} \in \mathcal{B}_{q_\omega}(\mathcal{U}_1 \cap \mathcal{U}_2)$ we can assume that $\varphi^s(V) \cap N_{q_\omega}^s \subset \mathcal{B}_{q_\omega}(\mathcal{U}_1 \cap \mathcal{U}_2)$.

So for all u_{q_ω} in the neighborhood $\varphi^s(V) \cap N_{q_\omega}^s$ of v_{q_ω}, we have:

$$\begin{aligned} F(u_{q_\omega}) &= \mathcal{B}_{q_\upsilon}\left(\mathcal{B}_{q_\omega}^{-1}(u_{q_\omega})\right) \\ &= \mathcal{B}_{q_\upsilon}\left(\pi((\varphi^s)^{-1}(u_{q_\omega}))\right) \\ &= \mathcal{B}_{q_\upsilon}\left(\pi(\Phi_{g_0}((\varphi^s)^{-1}(u_{q_\omega})))\right) \\ &= B_{q_\upsilon}(\Phi_{g_0}((\varphi^s)^{-1}(u_{q_\omega}))), \text{ since } \Phi_{g_0}((\varphi^s)^{-1}(u_{q_\omega})) \in U_2 \\ &= \varphi^s\left(\Phi_{\beta_{q_\upsilon}(\Phi_{g_0}((\varphi^s)^{-1}(u_{q_\omega})))}\left(\Phi_{g_0}((\varphi^s)^{-1}(u_{q_\omega}))\right)\right). \end{aligned}$$

An inspection of this formula (using the assumption (2.2) and the fact that the chart φ^s of Q^s can be seen as the restriction of a chart φ^{s-1} of Q^{s-1}) shows that, in a neighborhood of v_{q_ω}, F is C^1 as a map with values in $N^{s-1}_{q_\upsilon}$. Doing that for all $v_{q_\omega} \in \mathcal{B}_{q_\omega}(\mathcal{U}_1 \cap \mathcal{U}_2)$ we obtain the desired result. □

Now we consider the tangent map to F:

$$TF\colon \mathcal{B}_{q_\omega}(\mathcal{U}_1 \cap \mathcal{U}_2) \times N^s_{q_\omega} \to \mathcal{B}_{q_\upsilon}(\mathcal{U}_1 \cap \mathcal{U}_2) \times N^{s-1}_{q_\upsilon},$$
$$TF(u_{q_\omega}, v_{q_\omega}) := \left(F(u_{q_\omega}), D\,F(u_{q_\omega})(v_{q_\omega})\right).$$

As it was the case for B_q, TF makes sense and is continuous on the larger space $\mathcal{B}_{q_\omega}(\mathcal{U}_1 \cap \mathcal{U}_2) \times (N^s_{q_\omega})^W$. We will denote by

$$T^W F\colon \mathcal{B}_{q_\omega}(\mathcal{U}_1 \cap \mathcal{U}_2) \times (N^s_{q_\omega})^W \longrightarrow \mathcal{B}_{q_\upsilon}(\mathcal{U}_1 \cap \mathcal{U}_2) \times N^{s-1}_{q_\upsilon}$$

this extension. As in Lemma 4.2 we can show that for $d \in C^1_W(I, N^s_{q_x})$, $s > s_0 + 2$, such that $d(t) \in U$ for all $t \in I$, we have $F \circ d \in C^1_W(I, N^s_{q_y})$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(F \circ d)(t) = T^W_{d(t)} F(\dot{d}(t)). \tag{4.3}$$

Thus $T^W F$ takes values in $\mathcal{B}_{q_\upsilon}(\mathcal{U}_1 \cap \mathcal{U}_2) \times (N^s_{q_\upsilon})^W$, that is

$$T^W F\colon \mathcal{B}_{q_\omega}(\mathcal{U}_1 \cap \mathcal{U}_2) \times (N^s_{q_\omega})^W \longrightarrow \mathcal{B}_{q_\upsilon}(\mathcal{U}_1 \cap \mathcal{U}_2) \times (N^s_{q_\upsilon})^W.$$

Of course Theorem 4.6 and all previous remarks are valid for $F^{-1} = \mathcal{B}_{q_\omega} \circ (\mathcal{B}_{q_\upsilon})^{-1}$: $\mathcal{B}_{q_\upsilon}(\mathcal{U}_1 \cap \mathcal{U}_2) \subset N^s_{q_\upsilon} \longrightarrow \mathcal{B}_{q_\omega}(\mathcal{U}_1 \cap \mathcal{U}_2) \subset N^s_{q_\omega}$ and we can construct the tangent map $T^W(F^{-1})$. We have naturally the following property.

Theorem 4.7 *Under the hypothesis of the preceding theorem,*

$$T^W F\colon \mathcal{B}_{q_\omega}(\mathcal{U}_1 \cap \mathcal{U}_2) \times (N^s_{q_\omega})^W \longrightarrow \mathcal{B}_{q_\upsilon}(\mathcal{U}_1 \cap \mathcal{U}_2) \times (N^s_{q_\upsilon})^W$$

is an homeomorphism whose inverse is given by $T^W(F^{-1})$.

Proof As we remarked before, $T^W F$ is clearly continuous on $\mathcal{B}_{q_\omega}(\mathcal{U}_1 \cap \mathcal{U}_2) \times (N^s_{q_\omega})^W$. Let $d \in C^1_W(I, N^s_{q_x})$, so we have $F \circ d \in C^1_W(I, N^s_{q_y})$ and we can compute:

$$\begin{aligned}
\dot{d}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(F^{-1} \circ F \circ d)(t) \\
&= T^W_{F(d(t))}(F^{-1}) \left(\frac{\mathrm{d}}{\mathrm{d}t}(F \circ d)(t)\right) \quad \text{with formula (4.3), still valid for } F^{-1} \\
&= T^W_{F(d(t))}(F^{-1}) \left(T^W_{d(t)} F(\dot{d}(t))\right) \quad \text{with formula (4.3) again.}
\end{aligned}$$
□

Definition 4.8 For $s > s_0 + 1$ and I open in $\mathbb{R}$ we define the set $C^1_W(I, Q^s/G)$ of *weakly-differentiable* curves in Q^s/G as follows:

$c \in C^1_W(I, Q^s/G)$ if and only if for all $t_0 \in I$ there exists $\varepsilon > 0$ and $d \in C^1_W(]t_0 - \varepsilon, t_0 + \varepsilon[, Q^s)$ such that

$$c = \pi \circ d \quad \text{on} \quad]t_0 - \varepsilon, t_0 + \varepsilon[\ \subset I.$$

The following theorem shows that in a chart the notion of weakly-differentiable curves in Q^s/G coincides with the notion of weakly-differentiable curves in N_q^s given in Definition 4.4.

Theorem 4.9 *Let $s > s_0 + 2$ and I be open in $\mathbb{R}$. Then $c \in C^1_W(I, Q^s/G)$ if and only if for all charts we have $\mathcal{B}_q \circ c \in C^1_W(I', N_q^s)$, where I' is an open subset of I such that $\mathcal{B}_q \circ c$ is well-defined.*

Proof If $c \in C^1_W(I, Q^s/G)$ then, by definition, $c = \pi \circ d$ on $]t_0 - \varepsilon, t_0 + \varepsilon[$, where $d \in C^1_W(]t_0-\varepsilon, t_0+\varepsilon[, Q^s)$. So we obtain that $\mathcal{B}_q \circ c = \mathcal{B}_q \circ d \in C^1_W(I', N_q^s)$ by Lemma 4.2.

Conversely, let $t_0 \in I$, $\omega := c(t_0)$ and $q \in \pi^{-1}(\omega)$. We have $\mathcal{B}_q \circ c \in C^1_W(]t_0 - \varepsilon, t_0 + \varepsilon[, N_q^s)$. Let $d := (\varphi^s)^{-1} \circ \mathcal{B}_q \circ c$. Then $d \in C^1_W(]t_0 - \varepsilon, t_0 + \varepsilon[, Q^s)$ and $\pi \circ d = \pi \circ (\varphi^s)^{-1} \circ \mathcal{B}_q \circ c = c$. Doing that for each $t_0 \in I$ shows that $c \in C^1_W(I, Q^s/G)$. □

Now we are ready to construct the tangent space of the isotropy strata. This will be done using the notion of weakly-differentiable tangent curves at one point.

Definition 4.10 Let $\omega \in Q^s/G, s > s_0 + 2$, and $c_1, c_2 \in C^1_W(I, Q^s/G)$ such that $c_1(0) = c_2(0) = \omega$. We define the following relation:

$$c_1 \overset{\omega}{\sim} c_2 \Longleftrightarrow \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathcal{B}_q \circ c_1)(t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathcal{B}_q \circ c_2)(t)$$

where $\mathcal{B}_q$ is any chart at ω.

We emphasize that $\mathcal{B}_q \circ c_i$ is a curve in N_q^s but it is derived as a C^1 curve in N_q^{s-1} by Theorem 4.9.

Lemma 4.11 *In the preceding definition, the relation $\overset{\omega}{\sim}$ does not depend on the choice of the chart $\mathcal{B}_q$. Thus $\overset{\omega}{\sim}$ is an equivalence relation on the set $C^1_{W,\omega}(I, Q^s/G) := \{c \in C^1_W(I, Q^s/G) \mid c(0) = \omega\}, s > s_0 + 2$.*

Proof The proof is like the standard one but we shall do it because our curves are differentiable in a weaker sense. Let's suppose that $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathcal{B}_q \circ c_1)(t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathcal{B}_q \circ c_2)(t)$ and let $\mathcal{B}_r$ be another chart at ω. We have

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathcal{B}_q \circ c_1)(t) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathcal{B}_q \circ (\mathcal{B}_r)^{-1} \circ \mathcal{B}_r \circ c_1)(t) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (F \circ (\mathcal{B}_r \circ c_1))(t) \\ &= T^W_{\mathcal{B}_r(\omega)} F \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathcal{B}_r \circ c_1)(t) \right) \quad \text{because of formula (4.3).}\end{aligned}$$

Doing the same for c_2 and using the bijectivity of $T^W F$ we obtain that:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathcal{B}_r \circ c_1)(t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathcal{B}_r \circ c_2)(t).$$ □

Definition 4.12 Let $s > s_0 + 2$.

(i) Let $\omega \in Q^s/G$. The *weak tangent space* of Q^s/G at ω is defined by

$$T^W_\omega(Q^s/G) := C^1_{W,\omega}(I, Q^s/G)/ \overset{\omega}{\sim} .$$

We will denote by v_ω, $[c]_\omega$, or $\dot{c}(0)$ the elements of $T^W_\omega(Q^s/G)$.

(ii) The *weak tangent bundle* of Q^s/G is defined by

$$T^W(Q^s/G) := \bigcup_{\omega \in Q^s/G} T^W_\omega(Q^s/G).$$

(iii) Let $(\mathcal{U}, \mathcal{B}_{q_\omega})$ be a chart of Q^s/G. The *weak tangent map* of $\mathcal{B}_{q_\omega}$ is defined by

$$T^W\mathcal{B}_{q_\omega} : T^W(Q^s/G)|\mathcal{U} \longrightarrow \mathcal{B}_{q_\omega}(\mathcal{U}) \times (N^s_{q_\omega})^W,$$

$$T^W_\upsilon \mathcal{B}_{q_\omega}([c]_\upsilon) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathcal{B}_{q_\omega} \circ c)(t).$$

(iv) Let $(\mathcal{U}, \mathcal{B}_{q_\omega})$ be a chart of Q^s/G. The *weak tangent map* of $(\mathcal{B}_{q_\omega})^{-1}$ is defined by

$$T^W(\mathcal{B}_{q_\omega})^{-1} : \mathcal{B}_{q_\omega}(\mathcal{U}) \times (N^s_{q_\omega})^W \longrightarrow T^W(Q^s/G)|\mathcal{U},$$

$$T^W_{v_{q_\omega}}(\mathcal{B}_{q_\omega})^{-1}(v_{q_\omega}, w_{q_\omega}) := [(\mathcal{B}_{q_\omega})^{-1} \circ d]_{(\mathcal{B}_{q_\omega})^{-1}(v_{q_\omega})}$$

where $d \in C^1_W(I, N^s_{q_\omega})$ is such that $\dot{d}(0) = (v_{q_\omega}, w_{q_\omega})$.

(v) The *weak tangent map* of π is defined by

$$T^W\pi : T^W Q^s \longrightarrow T^W(Q^s/G), \quad T^W_r \pi(v_r) := [\pi \circ d]_{\pi(r)}$$

where $d \in C^1_W(I, Q^s)$ is such that $\dot{d}(0) = v_r$.

The properties of these weak tangent maps are summarized in the following statement.

Lemma 4.13 *We have the following.*

(i) $T^W(\mathcal{B}_{q_\omega})^{-1}$ *is well defined, that is, it does not depend on the choice of the curve d. Moreover:*

$$T^W\mathcal{B}_{q_\omega} \circ T^W(\mathcal{B}_{q_\omega})^{-1} = id.$$

(ii) $T^W\pi$ *is well-defined, that is, it does not depend on the choice of the curve d. Moreover for all charts* $\mathcal{B}_{q_\omega}$ *we have on* $T^W Q^s|U$:

$$T^W\mathcal{B}_{q_\omega} \circ T^W\pi = T^W(\mathcal{B}_{q_\omega} \circ \pi) = T^W B_{q_\omega}.$$

(iii) *Under the assumptions of Theorem 4.6 we have:*

$$T^W F = T^W\mathcal{B}_{q_v} \circ T^W(\mathcal{B}_{q_\omega})^{-1}.$$

Proof (i) Let $d(t)$ be a curve in $C^1_W(I, N^s_{q_x})$ such that $\dot{d}(0) = (v_{q_\omega}, w_{q_\omega}) \in \mathcal{B}_{q_\omega}(\mathcal{U}) \times (N^s_{q_\omega})^W$. Then we get

$$\begin{aligned}\left(T^W\mathcal{B}_{q_\omega} \circ T^W(\mathcal{B}_{q_\omega})^{-1}\right)(\dot{d}(0)) &= T^W_{(\mathcal{B}_{q_\omega})^{-1}(v_{q_\omega})}\mathcal{B}_{q_\omega}\left([(\mathcal{B}_{q_\omega})^{-1} \circ d]_{(\mathcal{B}_{q_\omega})^{-1}(v_{q_\omega})}\right)\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\mathcal{B}_{q_\omega}((\mathcal{B}_{q_\omega})^{-1}(d(t)))\right) = \dot{d}(0)\end{aligned}$$

by the definitions of $T^W\mathcal{B}_{q_\omega}$ and $T^W(\mathcal{B}_{q_\omega})^{-1}$. This shows that $T^W\mathcal{B}_{q_\omega} \circ T^W(\mathcal{B}_{q_\omega})^{-1} = id$. On the other hand, since $T^W_{v_{q_\omega}}(\mathcal{B}_{q_\omega})^{-1}(w_{q_\omega}) := [(\mathcal{B}_{q_\omega})^{-1} \circ d]_{(\mathcal{B}_{q_\omega})^{-1}(v_{q_\omega})} = \left(T^W\mathcal{B}_{q_\omega}\right)^{-1}(\dot{d}(0))$ we see that this expression depends only on $\dot{d}(0) = (v_{q_\omega}, w_{q_\omega})$ and not on the curve $d(t)$ itself.

(ii) For a curve $d(t)$ in $C^1_W(I, N^s_{q_\omega})$ such that $d(0) = r \in U$ we have

$$T^W_{\pi(r)}\mathcal{B}_{q_\omega}\left([\pi \circ d]_{\pi(r)}\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\mathcal{B}_{q_\omega}(\pi(d(t)))\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(B_{q_\omega}(d(t))\right) = T^W_r B_{q_\omega}(\dot{d}(0))$$

by the definition of $T^W\mathcal{B}_{q_\omega}$ and using Lemma 4.2. Since the right hand side does not depend on $d(t)$, it follows that $T^W\pi(v_r) := [\pi \circ d]_{\pi(r)}$ also does not depend on $d(t)$ satisfying $\dot{d}(0) = v_r$. In addition, the formula shows that $T^W\mathcal{B}_{q_\omega} \circ T^W\pi = T^W(\mathcal{B}_{q_\omega} \circ \pi) = T^W B_{q_\omega}$.
(iii) This follows directly from the definitions. □

Note that we can give $T^W_\omega(Q^s/G)$ the structure of a vector space as it is done in the standard case.

Theorem 4.14 *Let* $s > s_0 + 2$. *Then:*

(i) $T^W(Q^s/G)$ *is a topological manifold modeled on the space* $N^s_q \times (N^s_q)^W$ *with* $q \in Q^s$.
(ii) $T^W\mathcal{B}_{q_\omega}$, $T^W(\mathcal{B}_{q_\omega})^{-1}$, *and* $T^W\pi$ *are continuous maps.*

Proof (i) By the third part of the previous lemma, $(T^W\mathcal{U}, T^W\mathcal{B}_q)$ are topological charts of $T^W(Q^s/G)$, where $T^W\mathcal{U} := T^W(Q^s/G)|\mathcal{U}$. Hence $T^W(Q^s/G)$ is a topological manifold whose model is the space $N^s_q \times (N^s_q)^W$ with $q \in Q^s$. (ii) This is obvious. □

Definition 4.15 *Let $q \in Q^s, s > s_0 + 2$.*

(i) *The vertical subspace of $T_q^W Q^s$ is defined by*

$$V_q^W Q^s := \ker\left(T_q^W \pi\right)$$

(ii) *The horizontal subspace of $T_q^W Q^s$ relative to the metric γ is defined by*

$$H_q^W Q^s := \left(V_q^W Q^s\right)^{\perp}$$

where $\perp$ means the othogonal complement with respect to the inner product $\gamma(q)$.

The following lemma shows that our construction of tangent bundles in the situation of a non-smooth action is a good generalization of the standard situation.

Lemma 4.16 *Let $q \in Q^s$, $s > s_0 + 2$. Then we have:*

(i) $V_q^W Q^s = \{\xi_{Q^s}(q) \mid \xi \in \mathfrak{g}\}$.
(ii) $H_q^W Q^s = (N_q^s)^W = \{v_q \in T_q^W Q^s \mid \gamma(q)(v_q, \xi_{Q^s}(q)) = 0, \forall\, \xi \in \mathfrak{g}\}$.
(iii) *The projections associated to the decomposition $T_q^W Q^s = V_q^W Q^s \oplus H_q^W Q^s$ are given by*

$$\mathrm{ver}_q \colon T_q^W Q^s \longrightarrow V_q^W Q^s, \quad \mathrm{ver}_q(v_q) = \sum_{i=1}^n \gamma(q)\left(v_q, E_i(q)\right) E_i(q)$$

$$\mathrm{hor}_q \colon T_q^W Q^s \longrightarrow H_q^W Q^s, \quad \mathrm{hor}_q(v_q) := v_q - \mathrm{ver}_q(v_q),$$

where the basis $(e_1, \ldots, e_n)$ of $\mathfrak{g}$ is chosen such that $\gamma(q)\left(E_i(q), E_j(q)\right) = \delta_{ij}$.
(iv) *Let $\omega \in Q^s/G$ and $q_\omega \in \pi^{-1}(\omega)$. Then the map*

$$T_{q_\omega}^W \pi : H_{q_\omega}^W Q^s \longrightarrow T_\omega^W (Q^s/G)$$

is a continuous linear bijection. Its inverse, called the horizontal-lift, is denoted by

$$\mathrm{Hor}_{q_\omega} \colon T_\omega^W (Q^s/G) \longrightarrow H_{q_\omega}^W Q^s.$$

Proof (i) It suffices to prove that $V_q^W Q^s = \{\sum_{i=1}^n \lambda^i E_i(q) \mid \lambda^i \in \mathbb{R}\}$. Using the chart $\mathcal{B}_q$, (4.1), $\beta_q(q) = e$, and $T_q \varphi^s = id$, we have the following equivalences:

$$\begin{aligned} v_q \in V_q^W Q^s &\iff T_q^W \pi(v_q) = 0_{\pi(q)} \iff T_{\pi(q)}^W \mathcal{B}_q(T_q^W \pi(v_q)) = (\mathcal{B}_q(\pi(q)), 0_q) \\ &\iff T_q^W B_q(v_q) = (\mathcal{B}_q(\pi(q)), 0_q) \iff T_q \Phi_e(v_q) + T_e \Phi^q(T_q \beta_q(v_q))) = 0_q \\ &\iff v_q + T_e \Phi^q(T_q \beta_q(v_q)) = 0_q \iff v_q + T_e \Phi^q\left(\sum_{i=1}^n T_q \beta_q(v_q)^i e_i\right) = 0_q, \end{aligned}$$

where $T_q \beta_q(v_q)^i$ are the components of $T_q \beta_q(v_q)$ relative to the basis $(e_1, \ldots, e_n)$. Using that

$$T_e \Phi^q\left(\sum_{i=1}^n T_q \beta_q(v_q)^i e_i\right) = \sum_{i=1}^n T_q \beta_q(v_q)^i T_e \Phi^q(e_i) = \sum_{i=1}^n T_q \beta_q(v_q)^i E_i(q)$$

we obtain that $v_q = \sum_{i=1}^n \lambda^i E_i(q)$.

Conversely, for $v_q = \sum_{i=1}^n \lambda^i E_i(q)$, we have $T_q^W \pi(v_q) = \sum_{i=1}^n \lambda^i T_q^W \pi(E_i(q))$ and

$$T_q^W \pi(E_i(q)) = T_q^W \pi\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_{\exp(te_i)}(q)\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\pi \circ \Phi_{\exp(te_i)}\right)(q) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi(q) = 0.$$

(ii) By the definition and (i) we have

$$H_q^W Q^s := \left(V_q^W Q^s\right)^{\perp} = \{v_q \in T_q^W Q^s \mid \gamma(q)(v_q, \xi_{Q^s}(q)) = 0, \forall \xi \in \mathfrak{g}\}.$$

It remains to show that $H_q^W Q^s = (N_q^s)^W$. The map

$$T^W B_q \colon T^W Q^s | U \longrightarrow B_q(U) \times (N_q^s)^W \subset N_q^s \times (N_q^s)^W$$

is surjective by Lemma 4.13 (ii). Using formula (4.1) we find that $T^W B_q(v_q) = v_q$ for all $v_q \in H_q^W Q^s$, so we conclude that $v_q \in (N_q^s)^W$.

The other points are obvious. □

When the metric γ is G-invariant, we can define a metric $\widetilde{\gamma}$ on Q^s/G:

$$\widetilde{\gamma}(\omega)(u_\omega, v_\omega) := \gamma(q_\omega)(\mathrm{Hor}_{q_\omega}(u_\omega), \mathrm{Hor}_{q_\omega}(v_\omega))$$

where $\omega \in Q^s/G$, $u_\omega, v_\omega \in T_\omega^W(Q^s/G)$ and q_ω is any element in $\pi^{-1}(\omega)$. One can show that this expression does not depend on the choice of q_ω in $\pi^{-1}(\omega)$.

We now can show in which sense the (H)-orbit type set $(Q^s)_{(H)}$ can be seen as a manifold.

Consider a topological group G acting continuously on a topological space X by an action $\Phi \colon G \times X \longrightarrow X$. Let H be a subgroup of G. As before we can define the sets X^H, X_H, and $X_{(H)}$. Let $N(H) := \{g \in G \mid gHg^{-1} = H\}$ be the normalizer of H in G. We can consider the following well-defined twisted action of $N(H)$ on $G \times X_H$:

$$N(H) \times (G \times X_H) \longrightarrow (G \times X_H), \quad (h, (g, x)) \longmapsto (gh, \Phi_{h^{-1}}(x)).$$

The orbit space of this free action is denoted by $G \times_{N(H)} X_H$.

Remark that the continuous map $G \times X_H \longrightarrow X_{(H)}, \quad (g, x) \longmapsto \Phi_g(x)$ induces a well defined continuous bijection

$$G \times_{N(H)} X_H \longrightarrow X_{(H)}, \quad [(g, x)]_{N(H)} \longmapsto \Phi_g(x), \tag{4.4}$$

where $[(g, x)]_{N(H)}$ denotes the equivalence class of (g, x) relative to the twisted action.

We now consider the continuous map $X_H \longrightarrow X_{(H)}/G, \quad x \longmapsto [x]_G$, where $[x]_G$ denotes the equivalence class of x in $X_{(H)}$ relative to the action $\Phi \colon G \times X_{(H)} \longrightarrow X_{(H)}$. One can show that this map induces a continuous bijection

$$X_H/N(H) \longrightarrow X_{(H)}/G, \quad [x]_{N(H)} \longmapsto [x]_G, \tag{4.5}$$

where $[x]_{N(H)}$ denotes the equivalence class of x in X_H relative to the action Φ : $N(H) \times X_H \longrightarrow X_H$.

Remark that $X_H/(N(H)/H) = X_H/N(H)$, and that the action of $N(H)/H$ on X_H is free.

Applying the result (4.4) to the non-smooth action $\Phi \colon G \times Q^s \longrightarrow Q^s$ verifying the hypotheses of Sect. 2, we obtain a continuous bijection

$$G \times_{N(H)} Q_H^s \longrightarrow (Q^s)_{(H)}.$$

Since the twisted action of $N(H)$ on $G \times Q^s_H$ is free and verifies the hypotheses of Sect. 2, the orbit space $G \times_{N(H)} Q^s_H$ is a topological manifold (by Theorem 4.3) and all the results obtained in the present section are valid: we can define for $G \times_{N(H)} Q^s_H$ the weak tangent bundle and the weak differentiable curves. Using the previous continuous bijection, we can transport all these properties on $(Q^s)_{(H)}$.

In the same way, applying the result (4.5) to the non-smooth action $\Phi\colon G \times Q^s \longrightarrow Q^s$ verifying the hypotheses of Sect. 2, we obtain a continuous bijection

$$Q^s_H/(N(H)/H) \longrightarrow (Q^s)_{(H)}/G.$$

Since the action of $N(H)/H$ on Q^s_H is free and verifies the hypotheses of Sect. 2, the orbit space $Q^s_H/(N(H)/H)$ is a topological manifold and we can define the weak tangent bundle and the weak differentiable curves on it. Using the previous continuous bijection, we can transport all these properties on $(Q^s)_{(H)}/G$.

5 Differentiable functions on Q^s/G

Now we define a notion of differentiable functions on Q^s and Q^s/G that is compatible with the notion of weakly-differentiable curves in Q^s and in Q^s/G.

Definition 5.1 *Let $s > s_0 + 1$.*

(i) We define the set $C^1_W(Q^s)$ of *weakly-differentiable function* on Q^s as follows: $f \in C^1_W(Q^s)$ if and only if f is a real-valued function on Q^s which is C^1 relative to the $s-1$ differentiable structure on Q^s.

(ii) The set $C^1_W(Q^s/G)$ of *weakly-differentiable functions* on Q^s/G is defined as follows: $\varphi \in C^1_W(Q^s/G)$ if and only if φ is a real-valued function on Q^s/G such that $\varphi \circ \pi \in C^1_W(Q^s)$.

Note that for all $f \in C^1(Q^{s-1})$ we have $f|_{Q^s} \in C^1_W(Q^s)$. In this sense we can write the inclusion $C^1(Q^{s-1}) \subset C^1_W(Q^s)$. Moreover, for all $f \in C^1_W(Q^s)$ we can define the function $f \circ j_{(s,s-1)} \in C^1(Q^s)$. In this sense we can write the inclusion $C^1_W(Q^s) \subset C^1(Q^s)$. Thus we obtain

$$C^1(Q^{s-1}) \subset C^1_W(Q^s) \subset C^1(Q^s).$$

If $f \in C^1_W(Q^s)$ and $q \in Q^s$, by definition the differential

$$df(q)\colon T_qQ^s \longrightarrow \mathbb{R}$$

is a continuous linear map relative to the $s-1$ topology on T_qQ^s. Since Q^s is dense in Q^{s-1}, we can extend $df(q)$ to the map

$$d^Wf(q)\colon T^W_qQ^s \longrightarrow \mathbb{R}$$

which is linear and continuous on $T^W_qQ^s$ with the topology of T_qQ^{s-1}.

The next result shows that the notion of weakly-differentiable function in Definition 5.1 is compatible with the notion of weakly-differentiable curves in Definition 4.8. Note that for the first time, we use the density of the inclusion $j_{(s,s-1)}\colon Q^s \hookrightarrow Q^{s-1}$.

Theorem 5.2 *Let $s > s_0 + 1$ and $f \in C^1_W(Q^s)$. Then for all $q \in Q^s$ and $v_q \in T^W_q Q^s$ we have*

$$d^W f(q)(v_q) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(d(t))$$

where d is any curve in $C^1_W(I, Q^s)$ such that $d(0) = q$ and $\dot{d}(0) = v_q$.

Proof Let $d \in C^1_W(I, Q^s)$ be such that $\dot{d}(0) = v_q$. Since $f \circ d \in C^1(I, \mathbb{R})$ we have

$$\lim_{t\to 0} \frac{f(d(t)) - f(d(0))}{t} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(d(t)).$$

We shall show that

$$\lim_{t\to 0} \frac{f(d(t)) - f(d(0))}{t} = d^W f(q)(v_q).$$

Using a chart φ^s of Q^s at q, which is the restriction of a chart φ^{s-1} of Q^{s-1}, we have

$$\begin{aligned}
&\frac{f(d(t)) - f(d(0))}{t} - d^W f(q)(v_q) \\
&\quad= \frac{(f\circ(\varphi^s)^{-1})(\varphi^s(d(t))) - (f\circ(\varphi^s)^{-1})(\varphi^s(d(0)))}{t} \\
&\qquad - d^W(f\circ(\varphi^s)^{-1})(\varphi^s(d(0)))\left(T_{d(0)}\varphi^s(\dot{d}(0))\right) \\
&\quad= \frac{(f\circ(\varphi^s)^{-1})(\varphi^s(d(t))) - (f\circ(\varphi^s)^{-1})(\varphi^s(d(0)))}{t} \\
&\qquad - d(f\circ(\varphi^s)^{-1})(\varphi^s(d(0)))\left(\frac{\varphi^s(d(t)) - \varphi^s(d(0))}{t}\right) \\
&\qquad + d(f\circ(\varphi^s)^{-1})(\varphi^s(d(0)))\left(\frac{\varphi^s(d(t)) - \varphi^s(d(0))}{t}\right) \\
&\qquad - d^W(f\circ(\varphi^s)^{-1})(\varphi^s(d(0)))\left(T_{d(0)}\varphi^s(\dot{d}(0))\right) \\
&\quad= \frac{1}{t}\Big((f\circ(\varphi^s)^{-1})(\varphi^s(d(t))) - (f\circ(\varphi^s)^{-1})(\varphi^s(d(0))) \\
&\qquad - d(f\circ(\varphi^s)^{-1})(\varphi^s(d(0)))\left(\varphi^s(d(t)) - \varphi^s(d(0))\right)\Big) \\
&\qquad + d^W(f\circ(\varphi^s)^{-1})(\varphi^s(d(0)))\left(\frac{\varphi^s(d(t)) - \varphi^s(d(0))}{t} - T_{d(0)}\varphi^s(\dot{d}(0))\right).
\end{aligned}$$

By continuity of the linear map $d^W(f\circ(\varphi^s)^{-1})(\varphi^s(d(0)))$ with respect to the $s-1$ norm, the last term converges to 0. By weak-differentiability of $f\circ(\varphi^s)^{-1}$, for all $\varepsilon > 0$ we can choose $\delta > 0$ such that if $|t| < \delta$ the first term is less than

$$\varepsilon \left\| \frac{\varphi^s(d(t)) - \varphi^s(d(0))}{t} \right\|_{s-1}.$$

Since $\varphi^s \circ d \in C^1(I, T_q Q^{s-1})$, this expression converges to

$$\varepsilon \left\| \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi^s(d(t)) \right\|_{s-1}.$$

Thus we obtain that $\frac{f(d(t))-f(d(0))}{t}$ converges to $d^W f(q)(v_q)$. □

Inspired by the previous result we now define the differential of a function in $C^1_W(Q^s/G)$.

Definition 5.3 Let $s > s_0+2$ and $\varphi \in C^1_W(Q^s/G)$. For $\omega \in Q^s/G$ and $v_\omega \in T^W_\omega(Q^s/G)$, the *differential* of φ at ω in direction v_ω is defined by

$$d^W\varphi(\omega)(v_\omega) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi(c(t)),$$

where $c \in C^1_W(I, Q^s/G)$ is any curve such that $c(0) = \omega$ and $\dot{c}(0) = v_\omega$.

Theorem 5.4 *Let $s > s_0+2$ and $\varphi \in C^1_W(Q^s/G)$. Then the differential of φ at $\omega \in Q^s/G$ in direction $v_\omega \in T^W_\omega(Q^s/G)$ is well-defined, that is, it does not depend on the choice of the curve $c \in C^1_W(I, Q^s/G)$ satisfying $c(0) = \omega$ and $\dot{c}(0) = v_\omega$. Moreover, for $f := \varphi \circ \pi \in C^1_W(Q^s)$ and $q_\omega \in \pi^{-1}(\omega)$ we have*

$$d^W\varphi(\omega)(v_\omega) = d^W f(q_\omega)(\mathrm{Hor}_{q_\omega}(v_\omega))$$

Proof Let $c \in C^1_W(I, Q^s/G)$ satisfy $\dot{c}(0) = v_\omega$ such that $c = \pi \circ d \in C^1_W(I', Q^s)$, $I' \subset I$ (Definition 4.8 of $C^1_W(I, Q^s/G)$). Note that we have $\varphi \circ c = \varphi \circ \pi \circ d \in C^1(I, \mathbb{R})$ since $\varphi \circ \pi \in C^1_W(Q^s)$. So we can compute the derivative of $\varphi \circ c$. With $f := \varphi \circ \pi$ we have

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi(c(t)) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(d(t)) \\
&= d^W f(d(0))(\dot{d}(0)) \text{ by Theorem 5.2} \\
&= d^W f(d(0))(\mathrm{hor}_{d(0)}(\dot{d}(0))) \\
&= d^W f(d(0))(\mathrm{Hor}_{d(0)}(T^W_{d(0)}\pi(\dot{d}(0))) \\
&= d^W f(d(0))(\mathrm{Hor}_{d(0)}(\dot{c}(0))) \text{ by Definition 4.12 (v)} \\
&= d^W f(q_\omega)(\mathrm{Hor}_{q_\omega}(v_\omega)).
\end{aligned}$$

The third equality follows from the fact that $d^W f(q_\omega)$ vanishes on the vertical subspace. Namely, since $f = \varphi \circ \pi$ is G-invariant, that is, $f \circ \Phi_g = f$ for all $g \in G$, we obtain

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\Phi_{\exp(t\xi)}(q)) = d^W f(q)\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_{\exp(t\xi)}(q)\right) = d^W f(q)(\xi_{Q^s}(q)), \quad (5.1)$$

where we used that $t \longmapsto \Phi_{\exp(t\xi)}(q)$ is in $C^1_W(I, Q^s)$. □

6 Application to fluid dynamics

We consider the motion of an incompressible ideal fluid in a compact oriented Riemannian manifold M with boundary. It is well known that the configuration space is $\mathcal{D}^s_\mu(M)$, $s > \frac{\dim(M)}{2}+1$, the Hilbert manifold of volume preserving H^s-diffeomorphisms of M, and that the appropriate Lagrangian is given by the weak L^2 Riemannian metric

$$\langle\langle u_\eta, v_\eta \rangle\rangle_\eta = \int_M g(\eta(x))(u_\eta(x), v_\eta(x))\mu(x), \quad u_\eta, v_\eta \in T_\eta\mathcal{D}^s_\mu(M),$$

where g is the Riemannian metric on M and μ is the volume form induced by g. This Lagrangian is invariant under the two following commuting actions:

$$R\colon \mathcal{D}^s_\mu(M) \times T\mathcal{D}^s_\mu(M) \longrightarrow T\mathcal{D}^s_\mu(M),\ R(\eta, v_\xi) = R_\eta(v_\xi) := v_\xi \circ \eta$$

$$L\colon Iso^+ \times T\mathcal{D}^s_\mu(M) \longrightarrow T\mathcal{D}^s_\mu(M),\ L(i, v_\xi) = L_i(v_\xi) := Ti \circ v_\xi,$$

where $Iso^+ := Iso^+(M,g)$ denotes the group of Riemannian isometries of (M,g) which preserve the orientation. Since M is compact, it follows that Iso^+ is a compact Lie group of dimension $\leq \frac{n(n-1)}{2}$, $n = \dim(M)$.

We denote by $\mathfrak{iso}^+ := T_e Iso^+$ the Lie algebra of Iso^+. From Corollary 5.4 of Ebin [5], we know that Iso^+ can be seen as a submanifold of $\mathcal{D}^r(M)$, $r > \frac{\dim(M)}{2} + 1$. The tangent space at the identity of Iso^+, viewed as a submanifold of $\mathcal{D}^r(M)$, consists of smooth vector fields on M whose flows are curves in Iso^+, that is, it consists of the Killing vector fields:

$$\mathfrak{X}_K(M) = \{X \in \mathfrak{X}(M) \mid L_X g = 0\},$$

where $L_X g$ is the Lie derivative of g along X. The correspondence between $\mathfrak{iso}^+$ and $\mathfrak{X}_K(M)$ is given by

$$\mathfrak{iso}^+ \longrightarrow \mathfrak{X}_K(M),\ \ \xi \longmapsto X_\xi,$$

where $X_\xi(x) := T_e Ev_x(\xi)$ and $Ev_x : Iso^+ \longrightarrow M,\quad Ev_x(i) := i(x)$ is the evaluation map at x. Indeed, one can see that the flow of X_ξ is given by $\exp(t\xi)$, where exp is the exponential map of Iso^+.

Note that for all $i \in Iso^+$ we have $i_*\mu = \mu$, where μ is the volume form associated to g, so i is volume preserving and X_ξ is divergence free. For example, setting $M := \mathbb{B}^n$, the closed unit ball in $\mathbb{R}^n$, we have $Iso^+ = \mathrm{SO(n)}$ and for all $\xi \in \mathfrak{so}(\mathfrak{n})$ we have

$$X_\xi(x) = T_e Ev_x(\xi) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\xi)(x) = \xi x.$$

Our goal is to carry out the Poisson reduction by stages associated to these two commuting actions. For finite dimensional manifolds and Lie groups the reduction by stages procedure [12] guarantees that the two-stage reduction by the two commuting group actions yields the same result as the reduction by the product group. In our case this one step reduction by the product group $\mathcal{D}^s_\mu(M) \times Iso^+$ cannot be carried out because $\mathcal{D}^s_\mu(M)$ is not a Lie group and the action is not smooth in the usual sense. However, we shall see that the two step reduction, first by $\mathcal{D}^s_\mu(M)$ and then by the compact Lie group Iso^+, can be carried out in view of the general results proved above.

The result we shall obtain is a non-smooth generalization of the following theorem for proper smooth Lie group actions on finite dimensional manifolds in the physically relevant case of the Euler equations. Let $G \times M \to M$ be a smooth proper action of the Lie group G on the Poisson manifold $(M, \{\,,\})$. If (H) is an orbit type then $M_{(H)}/G$ is a smooth Poisson manifold. The Poisson structure is obtained by push forward of the natural quotient Poisson bracket on $M_H/N(H)$, where $N(H)$ is the normalizer of H in G. For details see Loja Fernandes et al. [11]. In our case we shall proceed in the following way. First we reduce $T\mathcal{D}^s_\mu(M)$ by the right action of $\mathcal{D}^s_\mu(M)$ and obtain $T\mathcal{D}^s_\mu(M)/\mathcal{D}^s_\mu(M) = \mathfrak{X}^s_{div}(M)$. The action of Iso^+ drops to a Poisson action on $\mathfrak{X}^s_{div}(M)$. Then we find the explicit Poisson bracket on the isotropy type manifolds $\mathfrak{X}^s_{div}(M)_H$ and on its quotient $\mathfrak{X}^s_{div}(M)_H/N(H)$, for any isotropy subgroup $H \subset Iso^+$. We close by presenting the reduced Euler equations on this quotient and discuss in what sense the flow is Poisson.

6.1 Reduction by $\mathcal{D}^s_\mu(M)$

Reduction by $\mathcal{D}^s_\mu(M)$ is well known [6] and leads to the Euler equations for an ideal incompressible fluid on the first reduced space $\mathfrak{X}^s_{div}(M) = T\mathcal{D}^s_\mu(M)/\mathcal{D}^s_\mu(M)$ consisting of H^s divergence free vector fields on M that are tangent to the boundary. The most fundamental fact is the existence of the smooth geodesic spray $\mathcal{S} \in \mathfrak{X}^{C^\infty}(T\mathcal{D}^s_\mu(M))$ of the weak Riemannian manifold $(\mathcal{D}^s_\mu(M), \langle\!\langle\, , \rangle\!\rangle)$. The following reduction theorem can by found in Ebin and Marsden [6].

Theorem 6.1 *Let $\eta(t) \subset \mathcal{D}^s_\mu(M), s > \frac{\dim(M)}{2} + 1$, be a curve in $\mathcal{D}^s_\mu(M)$ and let $u(t) := R_{\eta(t)^{-1}}(\dot\eta(t)) = \dot\eta(t) \circ \eta(t)^{-1} \in \mathfrak{X}^s_{div}(M)$. Then the following properties are equivalent.*

(i) *$\eta(t)$ is a geodesic of $(\mathcal{D}^s_\mu(M), \langle\!\langle\, , \rangle\!\rangle)$.*
(ii) *$V(t) := \dot\eta(t)$ is a solution of $\dot V(t) = \mathcal{S}(V(t))$.*
(iii) *$u(t)$ is a solution of the Euler equations*

$$\partial_t u(t) + \nabla_{u(t)} u(t) = -\operatorname{grad} p(t)$$

for some scalar function $p(t)\colon M \longrightarrow \mathbb{R}$ called the pressure.

Moreover the solution u of the Euler equation is in $C^0(I, \mathfrak{X}^s_{div}(M)) \cap C^1(I, \mathfrak{X}^{s-1}_{div}(M))$

Note that the Euler equations can be written as

$$\partial_t u(t) + P_e(\nabla_{u(t)} u(t)) = 0$$

where P_e denotes the projection on the first factor of the Hodge decomposition

$$\mathfrak{X}^r(M) = \mathfrak{X}^r_{div}(M) \oplus \operatorname{grad}(H^{r+1}(M)), \quad r \geq 0.$$

Denoting by $\pi_R\colon T\mathcal{D}^s_\mu(M) \longrightarrow \mathcal{D}^s_\mu(M),\ \ \pi_R(u_\eta) := u_\eta \circ \eta^{-1}$, the projection associated to the reduction by $\mathcal{D}^s_\mu(M)$ we obtain the following commutative diagram

$$\begin{tikzcd} T\mathcal{D}^s_\mu(M) \arrow[r, "F_t"] \arrow[d, "\pi_R"'] & T\mathcal{D}^s_\mu(M) \arrow[d, "\pi_R"] \\ \mathfrak{X}^s_{div}(M) \arrow[r, "\widetilde{F}_t"] & \mathfrak{X}^s_{div}(M). \end{tikzcd}$$

where F_t is the flow of $\mathcal{S}$ and $\widetilde{F}_t$ is the flow of the Euler equations. Formally, all these maps are Poisson, as is the case in the standard Poisson reduction procedure. However, since our manifolds are infinite dimensional, some difficulties arise. First, the symplectic form on $T\mathcal{D}^s_\mu(M)$ is only weak since the Lagrangian is given by a L^2 metric. Second, $\mathcal{D}^s_\mu(M)$ is not a Lie group since left multiplication and inversion are not smooth. Vasylkevych and Marsden [16] have resolved these difficulties by carefully analyzing the function spaces on which Poisson brackets are defined and carrying out a non-smooth Lie–Poisson reduction that takes into account all analytical difficulties. We recall below some results of this paper that we will use later on.

Poisson brackets on $T\mathcal{D}^s_\mu(M)$

For $F\colon T\mathcal{D}^s_\mu(M) \longrightarrow \mathbb{R}$ of class C^1, we define the horizontal partial derivative of F by

$$\frac{\partial F}{\partial \eta}\colon T\mathcal{D}^s_\mu(M) \to T^*\mathcal{D}^s_\mu(M)$$

such that

$$\frac{\partial F}{\partial \eta}(u_\eta)(v_\eta) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(\gamma(t)),$$

where $\gamma(t) \subset T\mathcal{D}^s_\mu(M)$ is a smooth path defined in a neighborhood of zero, with base point denoted by $\eta(t) \subset \mathcal{D}^s_\mu(M)$, satisfying the following conditions:

- $\gamma(0) = u_\eta$
- $\dot{\eta}(0) = v_\eta$
- γ is parallel, that is, its covariant derivative associated to the metric $\langle\!\langle\, ,\rangle\!\rangle$ vanishes.

The vertical partial derivative

$$\frac{\partial F}{\partial u}\colon T\mathcal{D}^s_\mu(M) \to T^*\mathcal{D}^s_\mu(M)$$

of F is defined as the usual fiber derivative, that is,

$$\frac{\partial F}{\partial u}(u_\eta)(v_\eta) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(u_\eta + t v_\eta).$$

These derivatives naturally induce corresponding functional derivatives relative to the weak Riemannian metric $\langle\!\langle\, ,\rangle\!\rangle$. The horizontal and vertical functional derivatives

$$\frac{\delta F}{\delta \eta}, \frac{\delta F}{\delta u}\colon T\mathcal{D}^s_\mu(M) \to T\mathcal{D}^s_\mu(M)$$

are defined by the equalities

$$\left\langle\!\!\left\langle \frac{\delta F}{\delta \eta}(u_\eta), v_\eta \right\rangle\!\!\right\rangle = \frac{\partial F}{\partial \eta}(u_\eta)(v_\eta) \qquad \text{and} \qquad \left\langle\!\!\left\langle \frac{\delta F}{\delta u}(u_\eta), v_\eta \right\rangle\!\!\right\rangle = \frac{\partial F}{\partial u}(u_\eta)(v_\eta)$$

for any $u_\eta, v_\eta \in T\mathcal{D}^s_\mu(M)$. Note that due to the weak character of $\langle\!\langle\, ,\rangle\!\rangle$, the existence of the functional derivatives is not guaranteed. But if they exist, they are unique.

We define, for $k \geq 1$ and $r, t > \frac{\dim(M)}{2} + 1$:

$$C^k_r(T\mathcal{D}^t_\mu(M)) := \left\{ F \in C^k(T\mathcal{D}^t_\mu(M)) \,\middle|\, \exists\, \frac{\delta F}{\delta \eta}, \frac{\delta F}{\delta u} : T\mathcal{D}^t_\mu(M) \longrightarrow T\mathcal{D}^r_\mu(M) \right\}.$$

With these definitions the Poisson bracket of $F, G \in C^k_r(T\mathcal{D}^t_\mu(M))$ is given by

$$\{F, G\}(u_\eta) = \left\langle\!\!\left\langle \frac{\delta F}{\delta \eta}(u_\eta), \frac{\delta G}{\delta u}(u_\eta) \right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \frac{\delta F}{\delta u}(u_\eta), \frac{\delta G}{\delta \eta}(u_\eta) \right\rangle\!\!\right\rangle. \tag{6.1}$$

Poisson brackets on $\mathfrak{X}^s_{div}(M)$

For $k \geq 1$ and $r \geq 0$ we define the set

$$C^k_r(\mathfrak{X}^s_{div}(M)) := \left\{ f \in C^k(\mathfrak{X}^s_{div}(M)) \,|\, \exists\, \delta f : \mathfrak{X}^s_{div}(M) \longrightarrow \mathfrak{X}^r_{div}(M) \right\}$$

where δf is the functional derivative of f with respect to the inner product $\langle\ ,\ \rangle := \langle\!\langle\ ,\ \rangle\!\rangle_{id}$, that is

$$\langle \delta f(u), v\rangle = Df(u)(v), \quad \forall\, u, v \in \mathfrak{X}^s_{div}(M).$$

For $k \geq 1$ and $r > \frac{\dim(M)}{2} + 1$, the Poisson bracket of $f, g \in C^k_r(\mathfrak{X}^s_{div}(M))$ is defined by

$$\{f, g\}_+(u) := \langle u, [\delta g(u), \delta f(u)]\rangle, \quad \forall u \in \mathfrak{X}^s_{div}(M). \tag{6.2}$$

The next theorem summarizes the principal results of Vasylkevych and Marsden [16].

Theorem 6.2 *Let $k \geq 1$.*

(i) *Let F_t be the flow of the geodesic spray $\mathcal{S}$ and let $t_1 \geq t_2 > \frac{\dim(M)}{2} + 1$. Then for all $G, H \in C^k_{t_2}(T\mathcal{D}^{t_1}_\mu(M))$ we have*

$$\{G \circ F_t, H \circ F_t\} = \{G, F\} \circ F_t$$

on $T\mathcal{D}^{t_1}_{\mu,D}$.

(ii) *Let $r > \frac{\dim(M)}{2} + 1$ satisfy $s + k \geq r$. Then for all $f, g \in C^k_r(\mathfrak{X}^s_{div}(M))$ we have*

$$\{f \circ \pi_R, g \circ \pi_R\}(u_\eta) = (\{f, g\}_+ \circ \pi_R)\,(u_\eta), \quad \forall u_\eta \in T\mathcal{D}^{s+k}_\mu(M).$$

(iii) *Let $\widetilde{F}_t$ be the flow of the Euler equations and let $r > 2$ be such that $s + k \geq r$. Then for all $f, g \in C^k_r(\mathfrak{X}^s_{div}(M))$ we have*

$$\{f \circ \tilde{F}_t, g \circ \tilde{F}_t\}_+(u) = \left(\{f, g\}_+ \circ \tilde{F}_t\right)(u), \quad \forall u \in \mathfrak{X}^{s+2k}_{div}(M).$$

In the next subsections we will carry out in a precise sense the second stage reduction, that is, the reduction by the group Iso^+.

6.2 Action of Iso^+ on $\mathfrak{X}^s_{div}(M)$

Recall that the action of Iso^+ on the tangent bundle of $\mathcal{D}^s_\mu(M)$ is

$$L\colon Iso^+ \times T\mathcal{D}^s_\mu(M) \longrightarrow T\mathcal{D}^s_\mu(M),\ \ L_i(v_\xi) = Ti \circ v_\xi.$$

Since R and L commute, L induces the action

$$l\colon Iso^+ \times \mathfrak{X}^s_{div}(M) \longrightarrow \mathfrak{X}^s_{div}(M),\ \ l_i(u) = i_* u.$$

Indeed, $l_i(u) := \pi_R(L_i(u)) = L_i(u) \circ i^{-1} = Ti \circ u \circ i^{-1} = i_* u$.

Remark that the action l is not smooth on $Iso^+ \times \mathfrak{X}^s_{div}(M)$. However, as we shall see below, l verifies all the hypothesis in Sect. 2. Thus the general theory developed in Sects. 3 to 5 is directly applicable to the present case.

In general, the action l is not free. Indeed, consider the particular case $M = \mathbb{D} \subset \mathbb{R}^2$, the closed unit disc, so we have $Iso^+ = \mathrm{SO}(2)$ and one sees that the vector field

$$z\colon \mathbb{D} \longrightarrow \mathbb{R}^2, \qquad z(a, b) = (-b, a)$$

is divergence free, tangent to the boundary, with isotropy group equal to SO(2).

We now proceed to the verification of the hypotheses of the Sect. 2. Consider the collection $\left\{\mathfrak{X}^s_{div}(M) \mid s > \frac{\dim(M)}{2}\right\}$ of Hilbert spaces. It is clear that the inclusions $j_{(r,s)}$

are smooth with dense range as well as their tangent maps. We endow $\mathfrak{X}^s_{div}(M)$ with the L^2 inner product

$$\langle u, v\rangle := \int_M g(x)(u(x), v(x))\mu(x)$$

which is the value at the identity of the weak Riemannian metric $\langle\langle\, ,\rangle\rangle$. Note that for all $i \in Iso^+$, we have

$$\langle l_i(u), l_i(v)\rangle = \langle u, v\rangle.$$

By Ebin [5] we know that the map

$$\mathcal{D}^r(M) \times \mathfrak{X}^s_{div}(M) \longrightarrow \mathfrak{X}^s_{div}(M), \quad (i, u) \longmapsto i_* u$$

is continuous for r sufficiently large; in Ebin [5], $\mathcal{D}^r(M)$ acts by pull back on Riemannian metrics and here on vector fields, but the proofs are similar. Since Iso^+ can be seen as a smooth submanifold of $\mathcal{D}^r(M)$, $r > \frac{\dim(M)}{2} + 1$, we obtain that

$$Iso^+ \times \mathfrak{X}^s_{div}(M) \longrightarrow \mathfrak{X}^s_{div}(M), \quad (i, u) \longmapsto i_* u$$

is continuous. Since Iso^+ is a compact Lie group, the action is proper.

We now check that l verifies the hypotheses (2.1) and (2.2). It is obvious that for all $i \in Iso^+$ the map

$$l_i\colon \mathfrak{X}^s_{div}(M) \longrightarrow \mathfrak{X}^s_{div}(M) \quad l_i(u) = i_* u$$

is smooth since it is linear and continuous. It remains to show that for all $u \in \mathfrak{X}^s_{div}(M), s > \frac{\dim(M)}{2} + 1$ the map

$$l^u\colon Iso^+ \longrightarrow \mathfrak{X}^{s-1}_{div}(M), \quad l^u(i) := i_* u$$

is C^1. By the proof of Proposition 3.4 of Ebin [5], the map

$$\mathcal{D}^r(M) \longrightarrow H^{s-1}(M, TM), \quad \eta \longmapsto u \circ \eta$$

is C^1 for r sufficiently large. Since Iso^+ is a smooth submanifold of $\mathcal{D}^r(M)$, the map

$$Iso^+ \longrightarrow H^{s-1}(M, TM), \quad i \longmapsto u \circ i$$

is C^1. Using that

$$Iso^+ \longrightarrow H^s(M, TM), \quad i \longmapsto Ti \circ u$$

and $i \longmapsto i^{-1}$ are smooth, we obtain the desired result.

The infinitesimal generator associated to $\xi \in \mathfrak{iso}^+$ is given by

$$\xi_{\mathfrak{X}^s_{div}(M)}(u) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} l_{\exp(t\xi)}(u) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\xi)_* u = -L_{X_\xi} u = [u, X_\xi], \tag{6.3}$$

where $X_\xi(x) = T_e Ev_x(\xi)$ is the Killing vector field generated by the flow $\exp(t\xi)$ and $[\,,]$ is the Jacobi–Lie bracket of vector fields. Remark that we have, as expected, $[u, X_\xi] \in \mathfrak{X}^{s-1}_{div}(M)$, since the Jacobi–Lie bracket of divergence free vector fields remains divergence free.

In the particular case $M = \mathbb{B}^n$ and $Iso^+ = \mathrm{SO(n)}$ we have $l_A(u)(x) = Au(A^{-1}x)$, so we compute directly

$$\xi_{\mathfrak{X}^s_{div}(\mathbb{B}^n)}(u)(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\xi)u(\exp(-t\xi)x) = \xi u(x) - Du(x)(\xi x).$$

Since $X_\xi(x) = \xi x$ and $DX_\xi(x) = \xi$ we obtain

$$\xi u(x) - Du(x)(\xi x) = DX_\xi(x)(u(x)) - Du(x)(X_\xi(x)) = [u, X_\xi](x),$$

so we recover the expression (6.3).

Now we can use the results in Sect. 3–5 with $Q^s = \mathfrak{X}^s_{div}(M)$ and $s_0 = \frac{\dim(M)}{2}$. We obtain the following results:

(i) *For $s > \frac{\dim(M)}{2} + 1$ and $H \subset Iso^+$ a closed subgroup, $\mathfrak{X}^s_{div}(M)_H$ is an open set in the vector subspace $\mathfrak{X}^s_{div}(M)^H$ of $\mathfrak{X}^s_{div}(M)$. As we did in the general case, we can define the weak tangent bundle $T^W\mathfrak{X}^s_{div}(M)$. Since $\mathfrak{X}^s_{div}(M)$ is a vector space, we use the notation $T^W\mathfrak{X}^s_{div}(M) = \mathfrak{X}^s_{div}(M) \times \mathfrak{X}^s_{div}(M)^W$.*

(ii) *Since the action $N(H)/H \times \mathfrak{X}^s_{div}(M)_H \longrightarrow \mathfrak{X}^s_{div}(M)_H$ is free and verifies the hypotheses in Sect. 2, we obtain that for $s > \frac{\dim(M)}{2} + 1$, $\mathfrak{X}^s_{div}(M)_H/N(H) = \mathfrak{X}^s_{div}(M)_H/(N(H)/H)$ is a topological manifold.*

(iii) *For $s > \frac{\dim(M)}{2} + 2$ we can define the weak tangent bundles*

$$T^W(\mathfrak{X}^s_{div}(M)),\ T^W(\mathfrak{X}^s_{div}(M)_H), \quad and \quad T^W(\mathfrak{X}^s_{div}(M)_H/N(H)),$$

the spaces of weakly-differentiable curves

$$C^1_W(I, \mathfrak{X}^s_{div}(M)),\ C^1_W(I, \mathfrak{X}^s_{div}(M)_H), \quad and \quad C^1_W(I, \mathfrak{X}^s_{div}(M)_H/N(H)),$$

the spaces of weakly-differentiable functions

$$C^1_W(\mathfrak{X}^s_{div}(M)),\ C^1_W(\mathfrak{X}^s_{div}(M)_H), \quad and \quad C^1_W(\mathfrak{X}^s_{div}(M)_H/N(H)),$$

the weak-tangent map $T^W\pi_H : T^W\mathfrak{X}^s_{div}(M)_H \longrightarrow T^W(\mathfrak{X}^s_{div}(M)_H/N(H))$, where $\pi_H : \mathfrak{X}^s_{div}(M)_H \longrightarrow \mathfrak{X}^s_{div}(M)_H/N(H)$ is the orbit map, the horizontal lift

$$\mathrm{Hor}^H_{u_\omega} : T^W_\omega(\mathfrak{X}^s_{div}(M)_H/N(H)) \longrightarrow T^W_{u_\omega}(\mathfrak{X}^s_{div}(M)_H),$$

and the vertical and horizontal projections hor^H_u and ver^H_u associated to the decomposition

$$T^W_u(\mathfrak{X}^s_{div}(M)_H) = H^W_u(\mathfrak{X}^s_{div}(M)_H) \oplus V^W_u(\mathfrak{X}^s_{div}(M)_H).$$

6.3 Dynamics on $(\mathfrak{X}^s_{div}(M), \{\,,\}_+)$

In this subsection we will show in which sense the action of Iso^+ is canonical with respect to the Lie–Poisson bracket $\{\,,\}_+$. We will prove existence and uniqueness of the Hamiltonian vector field associated to a Hamiltonian $h \in C^1_r(\mathfrak{X}^s_{div}(M))$ and then we will show that the Hamiltonian vector field associated to the Euler equations takes values in $T^W(\mathfrak{X}^s_{div}(M))$. Finally, we will see that the law of conservation of the isotropy is still valid in our case.

Lemma 6.3 *Let $i \in Iso^+$.*

(i) *For all $f \in C^1_r(\mathfrak{X}^s_{div}(M)), r, s \geq 0$ we have $f \circ l_i \in C^1_r(\mathfrak{X}^s_{div}(M))$ and*

$$\delta(f \circ l_i) = l_i^{-1} \circ \delta f \circ l_i.$$

(ii) *For all $u, v \in \mathfrak{X}^s_{div}(M), s > \frac{\dim(M)}{2} + 1$, we have*

$$[l_i(u), l_i(v)] = l_i([u, v]).$$

(iii) *The Hodge projector* $P_e\colon \mathfrak{X}^s(M) \longrightarrow \mathfrak{X}^s_{div}(M), s \geq 0$, *is* Iso^+*-invariant, that is,*

$$P_e \circ l_i = l_i \circ P_e.$$

Proof (i) By the chain rule and using that $f \in C^1_r(\mathfrak{X}^s_{div}(M))$ we have

$$\begin{aligned} D(f \circ l_i)(u)(v) &= Df(l_i(u))(Dl_i(u)(v)) \\ &= Df(l_i(u))(l_i(v)) \\ &= \langle \delta f(l_i(u)), l_i(v)\rangle \\ &= \langle l_i^{-1}(\delta f(l_i(u))), v\rangle. \end{aligned}$$

(ii) This is a consequence of the relation $i_*[u, v] = [i_*u, i_*v]$.
(iii) Decompose $u \in \mathfrak{X}^s_{div}(M)$ as $u = P_e(u) + \operatorname{grad} f$, so for all $i \in Iso^+$ we obtain $l_i(u) = l_i(P_e(u)) + l_i(\operatorname{grad} f)$. A direct computation using the chain rule gives the equality $l_i(\operatorname{grad} f) = \operatorname{grad}(f \circ i^{-1})$. Thus we can write $l_i(u) = l_i(P_e(u)) + \operatorname{grad}(f \circ i^{-1})$. Using that the Hodge decomposition is unique gives $P_e(l_i(u)) = l_i(P_e(u))$. □

The next theorem shows that the Iso^+-action is Poisson relative to $\{\,,\}_+$.

Theorem 6.4 *For all* $i \in Iso^+$ *and* $f, g \in C^1_r(\mathfrak{X}^s_{div}(M)), r, s > \frac{\dim(M)}{2} + 1$, *we have*

$$\{f \circ l_i, g \circ l_i\}_+ = \{f, g\}_+ \circ l_i$$

Proof By Lemma 6.3 and Iso^+-invariance of $\langle\,,\rangle$ we have

$$\begin{aligned} \{f \circ l_i, g \circ l_i\}_+(u) &= \langle u, [\delta(g \circ l_i)(u), \delta(f \circ l_i)(u)]\rangle \\ &= \langle u, [(l_i^{-1} \circ \delta g \circ l_i)(u), (l_i^{-1} \circ \delta g \circ l_i)(u)]\rangle \\ &= \langle u, l_i^{-1}([\delta g(l_i(u)), \delta f(l_i(u))])\rangle \\ &= \langle l_i(u), [\delta g(l_i(u)), \delta f(l_i(u))]\rangle \\ &= (\{f, g\}_+ \circ l_i)(u). \end{aligned}$$ □

Theorem 6.5 *Let* $h \in C^1_r(\mathfrak{X}^s_{div}(M)), r, s > \frac{\dim(M)}{2} + 1$. *Then there exists a unique Hamiltonian vector field* X_h *such that*

$$Df(u)(X_h(u)) = \{f, h\}_+(u), \quad \text{for all } f \in C^1_r(\mathfrak{X}^s_{div}(M)).$$

Moreover, X_h *is given by*

$$X_h(u) = -P_e(\nabla_{\delta h(u)} u + \nabla \delta h(u)^T \cdot u),$$

where ∇ *denotes the Levi–Civita covariant derivative and the upper index* T *denotes the transpose with respect to the Riemannian metric g. In particular, for* $h(u) = \frac{1}{2}\langle u, u\rangle$ *we have*

$$X_h(u) = -P_e(\nabla_u u)$$

and $X_h : \mathfrak{X}^s_{div}(M) \longrightarrow T^W\mathfrak{X}^s_{div}(M)$. *As usual, we think of the vector* $X_h(u)$ *also as* $X_h(u) = (u, -P_e(\nabla_u u))$ *when the need arises.*

Proof For $f,h \in C^1_r(\mathfrak{X}^s_{div}(M))$ integration by parts in the first term gives

$$\begin{aligned}\{f,h\}_+(u) &= \langle u, [\delta h(u), \delta f(u)]\rangle \\ &= \langle u, \nabla_{\delta h(u)}\delta f(u) - \nabla_{\delta f(u)}\delta h(u)\rangle \\ &= -\langle \nabla_{\delta h(u)}u, \delta f(u)\rangle - \langle \nabla\delta h(u)^T\cdot u, \delta f(u)\rangle \\ &= -\langle \delta f(u), \nabla_{\delta h(u)}u + \nabla\delta h(u)^T\cdot u\rangle \\ &= -\langle \delta f(u), P_e(\nabla_{\delta h(u)}u + \nabla\delta h(u)^T\cdot u)\rangle.\end{aligned}$$

Using that $Df(u)(X_h(u)) = \langle \delta f(u), X_h(u)\rangle$ we obtain by density the existence and uniqueness of $X_h(u) = -P_e(\nabla_{\delta h(u)}u + \nabla\delta h(u)^T\cdot u)$.

In particular, for the reduced Hamiltonian $h(u) = \frac{1}{2}\langle u,u\rangle$ we have $h \in C^1_s(\mathfrak{X}^s_{div}(M))$ and $\delta h(u) = u$. Thus $X_h(u) = -P_e(\nabla_u u + \nabla u^T\cdot u) = -P_e(\nabla_u u)$ since $2\nabla u^T\cdot u = \operatorname{grad}(g(u,u))$.

It remains to show that $X_h(u) \in T^W_u\mathfrak{X}^s_{div}(M)$, that is, there exists $d \in C^1_W(I, \mathfrak{X}^s_{div}(M))$ such that $d(0) = u$ and $\dot{d}(0) = P_e(\nabla_u u)$. It suffices to consider the curve $d(t) := P_e(T\eta(t)^T \circ u \circ \eta(t)) \in \mathfrak{X}^s_{div}(M)$ where $\eta(t)$ is a smooth curve in $\mathcal{D}^s_\mu(M)$ such that $\eta(0) = id$ and $\dot\eta(0) = u$. The fact that $d \in C^1_W(I, \mathfrak{X}^s_{div}(M))$ is a consequence of that fact that for $s > \frac{\dim(M)}{2} + 1$ the map

$$\omega_u\colon \mathcal{D}^s_\mu(M) \longrightarrow \mathfrak{X}^s(M),\ \omega_u(\eta) = u\circ\eta$$

is continuous and is C^1 as a map with values in $\mathfrak{X}^{s-1}(M)$ (see the proof of Proposition 3.4 in Ref. [5]). □

Note that for $u \in C^1_W(I, \mathfrak{X}^s_{div}(M)), s > \frac{\dim(M)}{2} + 1$, and $h(u) = \frac{1}{2}\langle u,u\rangle$ we have the following equivalent formulations of the Euler equations

(i)
$$\partial_t u(t) + \nabla_{u(t)}u(t) = -\operatorname{grad} p(t)$$

for some scalar function $p(t)\colon M \longrightarrow \mathbb{R}$,

(ii)
$$\partial_t u(t) = X_h(u(t)),$$

(iii) for all $f \in C^1_W(\mathfrak{X}^s_{div}(M))$ that admit a functional derivative $\delta f : \mathfrak{X}^s_{div}(M) \longrightarrow \mathfrak{X}^s_{div}(M)$ we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(u(t)) = \{f,h\}_+(u(t)).$$

It is well known, in the standard case, that the flow of invariant vector fields leaves the isotropy type submanifold invariant. Using existence and uniqueness of the solution of the Euler equations, we will show below that this property is still true for the vector field $X_h\colon \mathfrak{X}^s_{div}(M) \longrightarrow T^W\mathfrak{X}^s_{div}(M)$.

Theorem 6.6 *(Law of conservation of the isotropy) Let X_h be the Hamiltonian vector field associated to the Euler equation and let $\widetilde{F}_t$ be its flow. Then:*

(i) $X_h \circ l_i = l_i \circ X_h$ *and* $\widetilde{F}_t \circ l_i = l_i \circ \widetilde{F}_t$ *for all* $i \in Iso^+$,

(ii) *for all closed subgroups H of Iso^+, $X_h^H := X_h|_{\mathfrak{X}^s_{div}(M)_H}$ is a vector field on $\mathfrak{X}^s_{div}(M)_H$, that is*

$$X_h^H : \mathfrak{X}^s_{div}(M)_H \longrightarrow T^W \mathfrak{X}^s_{div}(M)_H,$$

and the flow of X_h^H is

$$\widetilde{F}_t^H := \widetilde{F}_t|_{\mathfrak{X}^s_{div}(M)_H} : \mathfrak{X}^s_{div}(M)_H \longrightarrow \mathfrak{X}^s_{div}(M)_H.$$

Proof (i) The invariance of X_h is a direct computation using Lemma 6.3. Let $u_0 \in \mathfrak{X}^s_{div}(M)$. We know that $\widetilde{F}_t(l_i(u_0))$ is an integral curve of X_h through $l_i(u_0)$. One can check, using the invariance of X_h, that the same is true for $l_i(\widetilde{F}_t(u_0))$. By uniqueness of the solutions of Euler equations we obtain that $\widetilde{F}_t \circ l_i = l_i \circ \widetilde{F}_t$, so we conclude that $\widetilde{F}_t(\mathfrak{X}^s_{div}(M)_H) \subset \mathfrak{X}^s_{div}(M)^H$. Using the bijectivity of the flow gives $\widetilde{F}_t(\mathfrak{X}^s_{div}(M)_H) \subset \mathfrak{X}^s_{div}(M)_H$. (ii) is a direct verification. □

Note that the preceding theorem remains valid for other Iso^+-invariant Hamiltonians h such that their corresponding motion equations

$$\partial_t u(t) + \nabla_{\delta h(u(t))} u(t) + \nabla \delta h(u(t))^T \cdot u(t) = -\operatorname{grad} p(t)$$

admit a unique integral curve $u \in C^1_W(I, \mathfrak{X}^s_{div}(M))$ for each initial condition.

6.4 Poisson brackets on $\mathfrak{X}^s_{div}(M)_H$ and $\mathfrak{X}^s_{div}(M)_H/N(H)$

In this subsection we will define a Poisson bracket $\{,\}_{\mathfrak{X}^s_H}$ on $\mathfrak{X}^s_{div}(M)_H$, and we shall show in which sense the inclusion map $i_H : \mathfrak{X}^s_{div}(M)_H \longrightarrow \mathfrak{X}^s_{div}(M)$ is a Poisson map. Then we will define a Poisson bracket $\{,\}_{\mathfrak{X}^s_H/N(H)}$ on $\mathfrak{X}^s_{div}(M)_H/N(H)$, and we shall show in which sense the projection map $\pi_H : \mathfrak{X}^s_{div}(M)_H \longrightarrow \mathfrak{X}^s_{div}(M)_H/N(H)$ is a Poisson map.

Definition 6.7 For $k \geq 1, s > \frac{\dim(M)}{2} + 1$, and $r \geq 0$ we define

$$C^k_r(\mathfrak{X}^s_{div}(M)_H) := \left\{ f \in C^k(\mathfrak{X}^s_{div}(M)_H) \mid \exists\, \delta f : \mathfrak{X}^s_{div}(M)_H \longrightarrow \mathfrak{X}^r_{div}(M)^H \right\}.$$

The *Poisson bracket* of $f, g \in C^k_r(\mathfrak{X}^s_{div}(M)_H), r > \frac{\dim(M)}{2} + 1$, is defined by

$$\{f, g\}_{\mathfrak{X}^s_H}(u) := \langle u, [\delta g(u), \delta f(u)] \rangle.$$

Theorem 6.8 *Let $k \geq 1$ and $r, s > \frac{\dim(M)}{2} + 1$.*

(i) *For all $f \in C^k_r(\mathfrak{X}^s_{div}(M))^H$, we have*

$$f \circ i_H \in C^k_r(\mathfrak{X}^s_{div}(M)_H) \quad \textit{and} \quad \delta(f \circ i_H) = \delta f \circ i_H.$$

(ii) *For all $f, g \in C^k_r(\mathfrak{X}^s_{div}(M))^H$, we have*

$$\{f \circ i_H, g \circ i_H\}_{\mathfrak{X}^s_H} = \{f, g\}_+ \circ i_H \;\; \textit{on} \;\; \mathfrak{X}^s_{div}(M)_H.$$

Proof (i) For all $u \in \mathfrak{X}^s_{div}(M)_H$ and for all $v \in \mathfrak{X}^s_{div}(M)^H$, we have:

$$D(f \circ i_H)(u)(v) = Df(i_H(u))(Di_H(u)(v)) = Df(i_H(u))(i_H(v)) = \langle \delta f(i_H(u)), i_H(v) \rangle.$$

We now show that $\delta f(i_H(u)) \in \mathfrak{X}^r_{div}(M)^H$. If $i \in H$, we have

$$\begin{aligned} l_i(\delta f(u)) &= l_i(\delta f(l_{i^{-1}}(u))), \quad \text{since } u \in \mathfrak{X}^s_{div}(M)_H \\ &= \delta(f \circ l_{i^{-1}})(u), \quad \text{by Lemma 6.3} \\ &= \delta f(u), \quad \text{since } f \in C^k_r(\mathfrak{X}^s_{div}(M))^H. \end{aligned}$$

Thus we obtain the existence of $\delta(f \circ i_H) = \delta f \circ i_H$.
(ii) Since $f \circ i_H, g \circ i_H \in C^k_r(\mathfrak{X}^s_{div}(M)_H)$ we can compute, for $u \in \mathfrak{X}^s_{div}(M)_H$,

$$\begin{aligned} \{f \circ i_H, g \circ i_H\}_{\mathfrak{X}^s_H}(u) &= \langle u, [\delta(g \circ i_H)(u), \delta(f \circ i_H)(u)]\rangle \\ &= \langle u, [\delta g(i_H(u)), \delta f(i_H(u))]\rangle \quad \text{by (i)} \\ &= \langle i_H(u), [\delta g(i_H(u)), \delta f(i_H(u))]\rangle \\ &= \{f, g\}_+(i_H(u)). \end{aligned}$$

□

In order to define the Poisson bracket on $\mathfrak{X}^s_{div}(M)_H/N(H)$, we will need the following subsets of $C^1_W(\mathfrak{X}^s_{div}(M)_H)$ and $C^1_W(\mathfrak{X}^s_{div}(M)_H/N(H))$.

Definition 6.9
(i) For $r \geq 0, s \geq 1$ we define the set

$$C^1_{Wr}(\mathfrak{X}^s_{div}(M)) := \left\{f \in C^1_W(\mathfrak{X}^s_{div}(M)) \mid \exists\, \delta f \colon \mathfrak{X}^s_{div}(M) \longrightarrow \mathfrak{X}^r_{div}(M)\right\}$$

where δf is the functional derivative of f with respect to $\langle\, , \rangle$, that is,

$$\langle \delta f(u), v\rangle = Df(u)(v), \quad \text{for all } u, v \in \mathfrak{X}^s_{div}(M).$$

This is possible since $C^1_W(\mathfrak{X}^s_{div}(M)) \subset C^1(\mathfrak{X}^s_{div}(M))$.
(ii) In a similar way we define

$$C^1_{Wr}(\mathfrak{X}^s_{div}(M)_H) := \left\{f \in C^1_W(\mathfrak{X}^s_{div}(M)_H) \mid \exists\, \delta f \colon \mathfrak{X}^s_{div}(M)_H \longrightarrow \mathfrak{X}^r_{div}(M)^H\right\}.$$

(iii) For $r \geq 0, s > \frac{\dim(M)}{2} + 2$ we define the set

$$C^1_{Wr}(\mathfrak{X}^s_{div}(M)_H/N(H)) := \left\{\varphi \in C^1_W(\mathfrak{X}^s_{div}(M)_H/N(H)) \mid \varphi \circ \pi_H \in C^1_{Wr}(\mathfrak{X}^s_{div}(M)_H)\right\},$$

where $\pi_H \colon \mathfrak{X}^s_{div}(M)_H \longrightarrow \mathfrak{X}^s_{div}(M)_H/N(H)$.

Recall that on $\mathfrak{X}^s_{div}(M)_H/N(H), s > \frac{\dim(M)}{2} + 2$, we can define a Riemannian metric

$$\gamma^H(\omega)(\xi_\omega, \eta_\omega) := \langle \mathrm{Hor}^H_{u_\omega}(\xi_\omega), \mathrm{Hor}^H_{u_\omega}(\eta_\omega)\rangle$$

where $\omega \in \mathfrak{X}^s_{div}(M)_H/N(H)$, $\xi_\omega, \eta_\omega \in T^W_\omega(\mathfrak{X}^s_{div}(M)_H/N(H))$, and u_ω is any element in $\pi_H^{-1}(\omega)$.

So for $\varphi \in C^1_{Wr}(\mathfrak{X}^s_{div}(M)_H/N(H))$ we have, using Theorem 5.4,

$$\begin{aligned} d^W\varphi(\omega)(\xi_\omega) &= D^W(\varphi \circ \pi_H)(u_\omega)(\mathrm{Hor}^H_{u_\omega}(\xi_\omega)) \\ &= \langle \delta(\varphi \circ \pi_H)(u_\omega), \mathrm{Hor}^H_{u_\omega}(\xi_\omega)\rangle \\ &= \langle \mathrm{Hor}^H_{u_\omega}(T^W_{u_\omega}\pi_H(\delta(\varphi \circ \pi_H)(u_\omega))), \mathrm{Hor}^H_{u_\omega}(\xi_\omega)\rangle \\ &= \gamma^H(\omega)(T^W_{u_\omega}\pi_H(\delta(\varphi \circ \pi_H)(u_\omega)), \xi_\omega). \end{aligned}$$

We conclude that any $\varphi \in C^1_{Wr}(\mathfrak{X}^s_{div}(M)_H/N(H))$ admits a functional derivative with respect to γ^H. It is given by

$$\delta\varphi\colon \mathfrak{X}^s_{div}(M)_H/N(H) \longrightarrow T^W(\mathfrak{X}^s_{div}(M)_H/N(H)),\ \delta\varphi(\omega) = T^W_{u_\omega}\pi_H(\delta(\varphi\circ\pi_H)(u_\omega)),$$

where u_ω is any element in $\pi_H^{-1}(\omega)$. Since $\varphi\circ\pi_H$ is $N(H)$-invariant, $\delta(\varphi\circ\pi_H)(\omega)$ is horizontal by (5.1), so we have

$$\delta(\varphi\circ\pi_H)(u_\omega) = \mathrm{Hor}^H_{u_\omega}(\delta\varphi(\omega)).$$

Lemma 6.10 *For all $u, v \in \mathfrak{X}^s_{div}(M), s > \frac{\dim(M)}{2}+1$, we have*

$$[u,v] \in \mathfrak{X}^s_{div}(M)^W$$

Proof It suffices to consider the curve $d(t) := \eta(t)^*v$ where $\eta(t)$ is a smooth curve in $\mathcal{D}^s_\mu(M)$ such that $\eta(0) = id$ and $\dot\eta(0) = u$. So we have $d \in C^1_W(I, \mathfrak{X}^s_{div}(M))$ and $\dot d(0) = L_u v = [u,v]$. □

Definition 6.11 The *Poisson bracket* of $\varphi, \psi \in C^1_{Wr}(\mathfrak{X}^s_{div}(M)_H/N(H)), r, s > \frac{\dim(M)}{2}+2$, is defined by

$$\{\varphi,\psi\}_{\mathfrak{X}^s_H/N(H)}(\omega) := \gamma^H(\omega)\left(S^H(\omega), [\![\delta\psi(\omega), \delta\varphi(\omega)]\!]^H\right),$$

where

(i) $S^H : \mathfrak{X}^s_{div}(M)_H/N(H) \longrightarrow T^W(\mathfrak{X}^s_{div}(M)_H/N(H))$ is the vector field on $\mathfrak{X}^s_{div}(M)_H/N(H)$ defined by $S^H(\omega) := T^W_{u_\omega}\pi_H(u_\omega)$ for any $u_\omega \in \pi_H^{-1}(\omega)$ and
(ii) $[\![\,,]\!]^H$ is the *reduced Lie bracket* defined by

$$[\![\xi_\omega, \eta_\omega]\!]^H := T^W_{u_\omega}\pi_H\left([\mathrm{Hor}^H_{u_\omega}(\xi_\omega), \mathrm{Hor}^H_{u_\omega}(\eta_\omega)]\right).$$

Note that since $\mathrm{Hor}^H_{u_\omega}(\delta\varphi(\omega)) = \delta(\varphi\circ\pi_H)(u_\omega) \in \mathfrak{X}^r_{div}(M), r > \frac{\dim(M)}{2}+2$, we have

$$[\mathrm{Hor}^H_{u_\omega}(\delta\varphi(\omega)), \mathrm{Hor}^H_{u_\omega}(\delta\psi(\omega))] \in \mathfrak{X}^r_{div}(M)^W$$

by Lemma 6.10. So $T^W_{u_\omega}\pi_H([\mathrm{Hor}_{u_\omega}(\delta\varphi(\omega)), \mathrm{Hor}_{u_\omega}(\delta\psi(\omega))])$ is well-defined.

Theorem 6.12 *For $s > \frac{\dim(M)}{2}+2$, the projection*

$$\pi_H\colon \mathfrak{X}^s_{div}(M)_H \longrightarrow \mathfrak{X}^s_{div}(M)_H/N(H)$$

is a Poisson map, that is, for all $\varphi, \psi \in C^1_{Wr}(\mathfrak{X}^s_{div}(M)_H/N(H)), r > \frac{\dim(M)}{2}+2$, we have

$$\{\varphi\circ\pi_H, \psi\circ\pi_H\}_{\mathfrak{X}^s_H} = \{\varphi,\psi\}_{\mathfrak{X}^s_H/N(H)}\circ\pi_H.$$

Proof Since $\varphi,\psi \in C^1_{Wr}(\mathfrak{X}^s_{div}(M)_H/N(H))$, we have $\varphi\circ\pi_H, \psi\circ\pi_H \in C^1_{Wr}(\mathfrak{X}^s_{div}(M)_H) \subset C^1_r(\mathfrak{X}^s_{div}(M)_H)$ by Definition 6.9 (iii). Note that $u \in T^W_u\mathfrak{X}^s_{div}(M)_H$ is horizontal, since we have $\langle u, \xi_{\mathfrak{X}^s_{div}(M)}(u)\rangle = \langle u, [u, X_\xi]\rangle = \langle u, \nabla_u X_\xi\rangle - \langle u, \nabla_{X_\xi}u\rangle = 0$. Indeed, integrating by parts we have $\langle u, \nabla_{X_\xi}u\rangle = -\langle\nabla_{X_\xi}u, u\rangle$ so $\langle u, \nabla_{X_\xi}u\rangle = 0$. Since

X_ξ is a Killing vector field, ∇X_ξ is skew symmetric, so $\langle u, \nabla_u X_\xi\rangle = 0$. This implies that $u = \mathrm{Hor}_u^H(S^H(\pi_H(u)))$. So for $u \in \mathfrak{X}^s_{div}(M)_H$ we have

$$\begin{aligned}
\{\varphi\circ\pi_H, \psi\circ\pi_H\}_{\mathfrak{X}^s_H}(u) &= \langle u, [\delta(\psi\circ\pi_H)(u), \delta(\varphi\circ\pi_H)(u)]\rangle \\
&= \langle \mathrm{Hor}_u^H(S^H(\pi_H(u))), [\delta(\psi\circ\pi_H)(u), \delta(\varphi\circ\pi_H)(u)]\rangle \\
&= \langle \mathrm{Hor}_u^H(S^H(\pi_H(u))), \mathrm{hor}_u^H([\delta(\psi\circ\pi_H)(u), \delta(\varphi\circ\pi_H)(u)])\rangle \\
&= \langle \mathrm{Hor}_u^H(S^H(\pi_H(u))), \mathrm{Hor}_u^H(T_u^W\pi_H([\delta(\psi\circ\pi_H)(u), \delta(\varphi\circ\pi_H)(u)]))\rangle \\
&= \gamma^H(\pi_H(u))\left(S^H(\pi_H(u)), T_u^W\pi_H([\mathrm{Hor}_u^H(\delta\psi(\pi_H(u))), \mathrm{Hor}_u^H(\delta\varphi(\pi_H(u)))])\right) \\
&= \{\varphi,\psi\}_{\mathfrak{X}^s_H/N(H)}(\pi_H(u))
\end{aligned}$$

by Definition 6.11. □

6.5 The reduced Euler equations

Recall that the Hamiltonian of the Euler equations is given by $h(u) = \frac{1}{2}\langle u, u\rangle$. We have $h \in C^1_{Ws}(\mathfrak{X}^s_{div}(M))$. The Hamiltonian vector field of h with respect to the Poisson bracket $\{\,,\}_+$, is given by (see (6.5))

$$X_h : \mathfrak{X}^s_{div}(M) \longrightarrow T^W\mathfrak{X}^s_{div}(M), \quad X_h(u) = (u, -P_e(\nabla_u u)).$$

By restriction to the isotropy type manifold, we have $h \in C^1_{Ws}(\mathfrak{X}^s_{div}(M)_H)$ and one can show that the Hamiltonian vector field with respect to the Poisson bracket $\{\,,\}_{\mathfrak{X}^s_H}$ is given by the restriction $X_h^H := X_h|\mathfrak{X}^s_{div}(M)_H$, that is,

$$X_h^H : \mathfrak{X}^s_{div}(M)_H \longrightarrow T^W\mathfrak{X}^s_{div}(M)_H, \quad X_h(u) = (u, -P_e(\nabla_u u)).$$

By $N(H)$-invariance, h induces a unique function $\mathfrak{h}^H : \mathfrak{X}^s_{div}(M)_H/N(H) \longrightarrow \mathbb{R}$, called the reduced Hamiltonian, such that $h = \mathfrak{h}^H \circ \pi_H$ on $\mathfrak{X}^s_{div}(M)_H$. By definition, we have $\mathfrak{h}^H \in C^1_{Ws}(\mathfrak{X}^s_{div}(M)_H/N(H))$ and a direct computation gives

$$\mathfrak{h}^H(\omega) = \frac{1}{2}\gamma^H(\omega)(S^H(\omega), S^H(\omega)).$$

Likewise, the Hamiltonian vector field X_h^H induces a unique vector field

$$X_{\mathfrak{h}^H} : \mathfrak{X}^s_{div}(M)_H/N(H) \longrightarrow T^W(\mathfrak{X}^s_{div}(M)_H/N(H))$$

such that $X_{\mathfrak{h}^H}\circ\pi_H = T^W\pi_H \circ X_h^H$ on $\mathfrak{X}^s_{div}(M)_H$. More precisely, we have the following result.

Lemma 6.13 *For $s > \frac{\dim(M)}{2} + 2$, $X_{\mathfrak{h}^H} : \mathfrak{X}^s_{div}(M)_H/N(H) \longrightarrow T^W(\mathfrak{X}^s_{div}(M)_H/N(H))$ is given by*

$$X_{\mathfrak{h}^H}(\omega) = -\mathcal{P}_e^H(\nabla^H_{S^H(\omega)} S^H(\omega)),$$

where

(i) *$\mathcal{P}_e^H : T^W(\mathfrak{X}^s(M)_H/N(H)) \longrightarrow T^W(\mathfrak{X}^s_{div}(M)_H/N(H))$ is the reduced Hodge projector given by*

$$\mathcal{P}_e^H(\xi_\omega) := T^W_{u_\omega}\pi_H(P_e(\mathrm{Hor}^H_{u_\omega}(\xi_\omega)))$$

for any $u_\omega \in \pi_H^{-1}(\omega)$ *and*

(ii) *for* $\xi_\omega, \eta_\omega \in T^W_\omega(\mathfrak{X}^s_{div}(M)_H/N(H))$,

$$\nabla^H_{\eta_\omega}\xi_\omega := T^W_{u_\omega}\pi_H(\nabla_{\mathrm{Hor}^H_{u_\omega}(\eta_\omega)}\mathrm{Hor}^H_{u_\omega}(\xi_\omega))$$

for any $u_\omega \in \pi_H^{-1}(\omega)$.

Proof For any $u \in \pi_H^{-1}(\omega)$ we have

$$\begin{aligned}
X_{\mathfrak{h}^H}(\omega) &= T^W_u\pi_H(X_h(u))\\
&= -T^W_u\pi_H(P_e(\nabla_u u))\\
&= -T^W_u\pi_H(P_e(\mathrm{hor}^H_u(\nabla_u u) + \mathrm{ver}^H_u(\nabla_u u)))\\
&= -T^W_u\pi_H(P_e(\mathrm{hor}^H_u(\nabla_u u)) + \mathrm{ver}^H_u(\nabla_u u))\\
&= -T^W_u\pi_H(P_e(\mathrm{Hor}^H_u(T^W_u\pi_H(\nabla_u u))))\\
&= -\mathcal{P}^H_e(T^W_u\pi_H(\nabla_u u))\\
&= -\mathcal{P}^H_e(T^W_u\pi_H(\nabla_{\mathrm{Hor}^H_u(S(\omega))}\mathrm{Hor}^H_u(S^H(\omega))))\\
&= -\mathcal{P}^H_e(\nabla^H_{S^H(\omega)}S^H(\omega)).
\end{aligned}$$

For the fourth equality we use that P_e is the identity on vertical vector fields, since they belong to $\mathfrak{X}^s_{div}(M)$. □

One can show as above that $X_{\mathfrak{h}^H}$ is the Hamiltonian vector field associated to $\mathfrak{h}^H$ with respect to $\{\,,\}_{\mathfrak{X}^s_H/N(H)}$, that is,

$$d^W\varphi(\omega)(X_{\mathfrak{h}^H}(\omega)) = \{\varphi, \mathfrak{h}^H\}_{\mathfrak{X}^s_H/N(H)}(\omega), \quad \text{for all } \varphi \in C^1_{Wr}(\mathfrak{X}^s_{div}(M)_H/N(H)).$$

Let $u(t)$ be an integral curve of the Euler equations. Thus we have $u \in C^1_W(I, \mathfrak{X}^s_{div}(M))$. If the initial condition $u(0) = u_0$ is in $\mathfrak{X}^s_{div}(M)_H$ then by the law of conservation of isotropy (Theorem 6.6) we have $u \in C^1_W(I, \mathfrak{X}^s_{div}(M)_H)$. Let $\omega := \pi_H \circ u$; then $\omega \in C^1_W(I, \mathfrak{X}^s_{div}(M)_H/N(H))$ and $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\omega(t) = X_{\mathfrak{h}^H}(\omega(t))$, that is,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\omega(t) = -\mathcal{P}^H_e(\nabla^H_{S^H(\omega(t))}S^H(\omega(t))).$$

These equations are called the *reduced Euler equations* on $\mathfrak{X}^s_{div}(M)_H/N(H)$.

Let $\widetilde{F}_t$ be the flow of the Euler equations on $\mathfrak{X}^s_{div}(M)$ and $\widetilde{F}^H_t$ the flow of the Euler equations on $\mathfrak{X}^s_{div}(M)_H$. Define $\widetilde{\widetilde{F}}_t{}^H(\omega) := \pi_H\left(\widetilde{F}^H_t(u_\omega)\right)$ where u_ω is such that $\pi_H(u_\omega) = \omega$. Then $\widetilde{\widetilde{F}}_t{}^H(\omega)$ is the flow of the reduced Euler equations on $\mathfrak{X}^s_{div}(M)_H/N(H)$, that is, $\omega(t) := \widetilde{\widetilde{F}}_t{}^H(\omega_0)$ is the integral curve through ω_0. Moreover, we have the commutative diagram

$$\begin{tikzcd}
\mathfrak{X}^s_{div}(M)_H \arrow[r, "F^H_t"] \arrow[d, "\pi_H"'] & \mathfrak{X}^s_{div}(M)_H \arrow[d, "\pi_H"] \\
\mathfrak{X}^s_{div}(M)_H/N(H) \arrow[r, "F_t{}^H"] & \mathfrak{X}^s_{div}(M)_H/N(H).
\end{tikzcd}$$

We already know that $\widetilde{F}_t^H$ and π_H are Poisson maps in the precise sense given in Theorems 6.2 and 6.12. In the following theorem we show in which sense $\widetilde{\widetilde{F}}_t{}^H$ is a Poisson map.

Theorem 6.14 *For all* $\varphi, \psi \in C^1_{Wr}(\mathfrak{X}^s_{div}(M)_H/N(H)), r, s > \frac{\dim(M)}{2} + 2$, *such that* $s + 1 \geq r$ *we have*

$$\{\varphi, \psi\}_{\mathfrak{X}^s_H/N(H)} \circ \widetilde{\widetilde{F}}_t{}^H = \{\varphi \circ \widetilde{\widetilde{F}}_t{}^H, \psi \circ \widetilde{\widetilde{F}}_t{}^H\}_{\mathfrak{X}^s_H/N(H)}$$

on $\mathfrak{X}^{s+2}_{div}(M)_H/N(H)$.

Proof For $\omega \in \mathfrak{X}^{s+2}_{div}(M)_H/N(H)$ we have for any $u_\omega \in \pi_H^{-1}(\omega)$

$$\begin{aligned}
\{\varphi, \psi\}_{\mathfrak{X}^s_H/N(H)}(\widetilde{\widetilde{F}}_t{}^H(\omega)) &= \{\varphi, \psi\}_{\mathfrak{X}^s_H/N(H)}(\pi_H(\widetilde{F}_t^H(u_\omega))) \\
&= \{\varphi \circ \pi_H, \psi \circ \pi_H\}_{\mathfrak{X}^s_H}(\widetilde{F}_t^H(u_\omega)) \quad \text{by Theorem 6.12} \\
&= \{\varphi \circ \pi_H \circ \widetilde{F}_t^H, \psi \circ \pi_H \circ \widetilde{F}_t^H\}_{\mathfrak{X}^s_H}(u_\omega) \\
&= \{\varphi \circ \widetilde{\widetilde{F}}_t{}^H \circ \pi_H, \psi \circ \widetilde{\widetilde{F}}_t{}^H \circ \pi_H\}_{\mathfrak{X}^s_H}(u_\omega) \quad \text{by the commutative diagram} \\
&= \{\varphi \circ \widetilde{\widetilde{F}}_t{}^H, \psi \circ \widetilde{\widetilde{F}}_t{}^H\}_{\mathfrak{X}^s_H/N(H)}(\pi_H(u_\omega)) \quad \text{by Theorem 6.12} \\
&= \{\varphi \circ \widetilde{\widetilde{F}}_t{}^H, \psi \circ \widetilde{\widetilde{F}}_t{}^H\}_{\mathfrak{X}^s_H/N(H)}(\omega).
\end{aligned}$$

For the third equality we use Theorem 6.2 (iii) which is applicable since we assumed that $s + 1 \geq r$ and $u_\omega \in \mathfrak{X}^{s+2}_{div}(M)_H$.

For the fifth equality we use that $\varphi \circ \widetilde{\widetilde{F}}_t{}^H \in C^1_{Wr}(\mathfrak{X}^{s+2}_{div}(M)_H)$. Namely, from Vasylkevych and Marsden [16] we know that for all $f \in C^1_r(\mathfrak{X}^s_{div}(M))$ we have $f \circ \widetilde{F}_t \in C^1_r(\mathfrak{X}^{s+1}_{div}(M))$. So we obtain that $f \circ \widetilde{F}_t^H|_{\mathfrak{X}^{s+2}_{div}(M)_H} \in C^1_{Wr}(\mathfrak{X}^{s+2}_{div}(M)_H)$ and $\varphi \circ \widetilde{\widetilde{F}}_t{}^H \circ \pi_H = \varphi \circ \pi_H \circ \widetilde{F}_t^H \in C^1_{Wr}(\mathfrak{X}^{s+2}_{div}(M)_H)$ because $\varphi \circ \pi_H \in C^1_{Wr}(\mathfrak{X}^s_{div}(M)_H) \subset C^1_r(\mathfrak{X}^s_{div}(M)_H)$. Thus, by definition, we find that $\varphi \circ \widetilde{\widetilde{F}}_t{}^H \in C^1_{Wr}(\mathfrak{X}^{s+2}_{div}(M)_H)$. □

Finally we obtain the following commuting diagram in which all maps are Poisson in the precise sense given in Theorems 6.2, 6.8, 6.12, and 6.14.

$$\begin{tikzcd}
T\mathcal{D}^s_\mu(M) \arrow[r, "\widetilde{F}_t"] \arrow[d, "\pi_R"'] & T\mathcal{D}^s_\mu(M) \arrow[d, "\pi_R"] \\
\mathfrak{X}^s_{div}(M) \arrow[r, "\widetilde{F}_t"] & \mathfrak{X}^s_{div}(M) \\
\mathfrak{X}^s_{div}(M)_H \arrow[r, "\widetilde{F}_t^H"] \arrow[u, "i_H"] \arrow[d, "\pi_H"'] & \mathfrak{X}^s_{div}(M)_H \arrow[u, "i_H"'] \arrow[d, "\pi_H"] \\
\mathfrak{X}^s_{div}(M)_H/N(H) \arrow[r, "\widetilde{\widetilde{F}}_t{}^H"] & \mathfrak{X}^s_{div}(M)_H/N(H).
\end{tikzcd}$$

The same results can be obtained for the averaged Euler equations by using Gay-Balmaz and Ratiu [7] instead of Vaskylkevych and Marsden [16].

Acknowledgements We thank J. Marsden for suggesting the problem and for several discussions during the elaboration of the paper. Our thanks to P. Michor for conversations regarding infinite dimensional geometry. The first author was fully supported by a doctoral fellowship of the EPFL. The second author acknowledges the partial support of the Swiss NSF.

References

1. Abraham, R., Marsden, J.E.: Foundations of Mechanics. Benjamin-Cummings Publ. Co, Updated 1985 version, reprinted by Perseus Publishing, 2nd edn (1978)
2. Bourbaki, N.: Groupes et algèbres de Lie. Chapitres 2 et 3. Hermann, Paris (1972)
3. Chernoff, P.R., Marsden, J.E.: Properties of Infinite Dimensional Hamiltonian Systems, vol. 425 of Lecture Notes in Math. Springer, New York (1974)
4. Cushman, R.H., Bates, L.M.: Global Aspects of Classical Integrable Systems. Birkäuser, Boston (1997)
5. Ebin, D.G.: The manifold of Riemannian metrics. Proc. Sympos. Pure Math. **15**, 11–40 (1968)
6. Ebin, D.G., Marsden, J.E.: Groups of diffeomorphisms and the motion of an incompressible fluid. Ann. Math. **92**, 102–163 (1970)
7. Gay-Balmaz, F., Ratiu, T.S.: The Lie–Poisson structure of the LAE-α equation. Dyn. PDE **2**(1), 25–57 (2005)
8. Isenberg, J., Marsden, J.E.: A slice theorem for the space of solutions of Einstein equations. Phys. Rep. **89**, 179–222 (1982)
9. Karcher, H.: On the Hilbert manifold $H_1(S^1, M)$ of closed curves. Comm. Pure Appl. Math. **23**, 201–219 (1970)
10. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, vol. I. Interscience Publishers, a division of John Wiley & Sons, New York (1963)
11. Loja Fernandes, R., Ortega, J.-P., Ratiu, T.S.: Momentum maps in Poisson geometry, preprint (2006)
12. Marsden, J.E., Misiolek, G., Ortega, J.-P., Perlmutter, M., Ratiu, T.S.: Hamiltonian Reduction by Stages, Lecture Notes in Mathematics (2006) (to appear)
13. Meyers, S.B., Steenrod, N.: The group of isometries of a Riemannian manifold. Ann. Math. **40**, 400–456 (1939)
14. Ortega, J-P., Ratiu, T.S.: Momentum Maps and Hamiltonian Reduction. Progress in Mathematics, vol. 222, Birkäuser, Boston (2004)
15. Palais, R.S.: On the existence of slices for actions of noncompact Lie groups. Ann. Math. **73**, 295–323 (1968)
16. Vasylkevych, S., Marsden, J.E.: The Lie–Poisson structure of the Euler equations of an ideal fluid. Dyn PDE **2**(4), 281–300 (2005)